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RELATIVE PERTURBATION TECHNIQUES FOR SINGULAR VALUE PROBLEMS*

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Abstract. We present a technique for deriving bounds on the *relative* change in the singular values of a real matrix (or the eigenvalues of a real symmetric matrix) due to a perturbation, as well as bounds on the angles between the unperturbed and perturbed singular vectors (or eigenvectors). In particular we consider the class of perturbations δB for which $B+\delta B = D_L B D_R$ for some nonsingular matrices D_L and D_R. This class includes componentwise relative perturbations of a bidiagonal or biacyclic matrix, and perturbations that annihilate the off-diagonal block in a block triangular matrix. We show how many existing relative perturbation and deflation bounds can be derived from results for this general class of perturbations. We also present some new relative perturbation and deflation results for the singular values and vectors of biacyclic, triangular, and shifted triangular matrices.

Key words. singular value decomposition, componentwise relative perturbations, deflation criteria, bidiagonal matrices, biacyclic matrices

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1. Introduction.

We present a technique for deriving bounds on the *relative* change in the singular values of a real matrix (or the eigenvalues of a real symmetric matrix) due to a perturbation, as well as bounds on the angles between the unperturbed and perturbed singular vectors (or eigenvectors).

Convergence and deflation criteria based on such relative perturbation results can be incorporated into QR-type algorithms for computing the singular values and singular vectors of dense or banded matrices, leading to highly accurate and efficient implementations. For instance, the implementation of the Golub-Kahan algorithm in [5, 6] computes the smallest singular values of a bidiagonal matrix to high relative accuracy, and the associated singular vectors with an error bound inversely proportional to the relative gap; and the implementation of Rutishauser's QD algorithm in [11] computes all of the singular values of a bidiagonal matrix to high relative accuracy. In general, deflation criteria enhance efficiency by breaking the problem up into smaller subproblems, which also makes them suitable for the design of accurate parallel divide-and-conquer algorithms [3].

In the context of the singular value decomposition of a real matrix B, we consider the class of perturbations δB for which $B + \delta B = D_L B D_R$ for some nonsingular matrices D_L and D_R (cf. [17], [1, p. 770]). This is the class of all perturbations that do not change the rank of B. In particular, it includes componentwise relative perturbations of bidiagonal or biacyclic matrices, and perturbations that annihilate the off-diagonal block in a block triangular matrix.

We show how many existing relative perturbation bounds for singular values and vectors of bidiagonal matrices can be derived from general results for this class of perturbations. We also present new relative perturbation results for the singular values

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and vectors of biacyclic, triangular, and shifted triangular matrices. Surprisingly, all of these results can be derived from traditional absolute perturbation results.

The standard absolute perturbation result says that the eigenvalues of the real symmetric matrices M and $M+\delta M$ satisfy

$$|\lambda_i[M+\delta M]-\lambda_i[M]| \le \|\delta M\|, \tag{1.1}$$

where $\lambda_i[X]$ denotes the i^{th} largest eigenvalue of X and $\|\cdot\|$ denotes the two-norm (see [21, Fact 1-11]). Moreover, if w is an eigenvector of M corresponding to $\lambda_i[M]$ and w' is an eigenvector of $M+\delta M$ corresponding to *some* eigenvalue λ', i.e.,

$$Mw = \lambda_i[M]w \qquad \text{and} \qquad (M+\delta M)w' = \lambda' w',$$

then the $\sin\theta$-Theorem [4] implies that

$$|\sin\theta| \le \frac{\|\delta M\|}{\mathrm{gap}_i}, \tag{1.2}$$

where $0 \le \theta \le \pi/2$ is the angle between the spaces spanned by w and by w' and

$$\mathrm{gap}_i \equiv \min_{j\ne i} |\lambda_j[M]-\lambda'|$$

is the *absolute gap* (see [21, Fact 1-12]).

These absolute perturbation results are used in §2 in the context of the eigenproblem to derive two relative perturbation results for perturbations of the form $M+\delta M = D^T M D$, where D is a nonsingular matrix. Since this is the class of congruence transformations of M, the inertia is preserved, so that $\lambda_i[M]$ and $\lambda_i[M+\delta M]$ have the same sign (or are both zero).

The eigenvalue result

$$|\lambda_i[M+\delta M]-\lambda_i[M]| \le |\lambda_i[M]|\,\|D^T D - I\|$$

is relative because it bounds the relative perturbation in the eigenvalue. The factor $\|D^T D - I\|$ represents the deviation of D from orthogonality.

The eigenvector result

$$\sin\theta \le \frac{\delta}{\rho_i[M]-\gamma} + \beta,$$

provided that[1] $\rho_i[M] > \gamma$, where

$$\beta \equiv \|D-I\|, \qquad \gamma \equiv \|D^T D - I\|, \qquad \delta \equiv \|D^T D\|\,\|D^{-T}D^{-1}-I\|,$$

and[2]

$$\rho_i[M] \equiv \min_{j\ne i} \frac{|\lambda_j[M]-\lambda_i[M]|}{|\lambda_i[M]|},$$

is relative in the sense that it depends on the *relative gap* $\rho_i[M]$. Here γ and δ represent the deviation of D from orthogonality, and β represents the rotation of the eigenvectors (note that $\gamma=\delta=0$ if D is orthogonal).

In §3 we derive analogous results for singular values and singular vectors by formulating the singular value problem as a symmetric eigenvalue problem. The remaining sections are devoted to applying these results to special classes of matrices and perturbations. In §4 we consider componentwise relative perturbations of symmetric tridiagonal matrices with zero diagonal elements, bidiagonal matrices, and biacyclic matrices.

[1] Note that that $\lambda_i[M]$ must be simple; otherwise $\rho_i[M]=0$, and this condition is not satisfied.
[2] We adopt the convention that $\rho_i[M]=+\infty$ when $\lambda_i[M]=0$, in which case $\delta/(\rho_i[M]-\gamma)=0$.

We obtain simple proofs of many of the relative perturbation results in [1, 5, 8, 6] and solve an open problem in [8]. In §5 we consider perturbations that introduce zeros into a block triangular or bidiagonal matrix. We give simple proofs of many of the deflation results in [2, 5, 6, 16, 20] and strengthen the deflation result in [11]. In §6 we discuss some further applications.

Notation. All matrices are real. The identity matrix is denoted by I, and its k^{th} column by e_k. The two-norm is denoted by $\|\cdot\|$, and the one-norm by $\|\cdot\|_1$.

Let M be a $n \times n$ symmetric matrix. The eigenvalues $\{\lambda_i[M]\}_{i=1}^{n}$ of M are numbered in decreasing order so that $\lambda_1[M] \geq \lambda_2[M] \geq \cdots \geq \lambda_n[M]$. Let N be a $m\times n$ matrix and let $p = \min\{m, n\}$. The singular values $\{\sigma_i[N]\}_{i=1}^{p}$ of N are numbered in decreasing order so that $\sigma_1[N] \geq \sigma_2[N] \geq \cdots \geq \sigma_p[N] \geq 0$. Also, $\sigma_{max}[N] = \sigma_1[N]$ and $\sigma_{min}[N] = \sigma_p[N]$.

By convention, A and $A + \delta A$ denote $n \times n$ symmetric matrices; λ_i and λ_i' denote the i^{th} eigenvalues of A and $A + \delta A$, respectively; w_i and w_i' denote unit eigenvectors associated with λ_i and λ_i', respectively; θ_i denotes the angle ($0 \leq \theta_i \leq \pi/2$) between the subspaces spanned by w_i and by w_i'; and[3]

$$\rho_i \equiv \min_{j\neq i} \frac{|\lambda_j - \lambda_i|}{|\lambda_i|}$$

denotes the relative gap for the i^{th} eigenvalue of A.

Similarly, B and $B + \delta B$ denote $m \times n$ general matrices; σ_i and σ_i' denote the i^{th} singular values of B and $B + \delta B$, respectively; u_i and u_i' denote unit left singular vectors associated with σ_i and σ_i', respectively; v_i and v_i' denote the corresponding right singular vectors; θ_i^u denotes the angle ($0 \leq \theta_i^u \leq \pi/2$) between the subspaces spanned by u_i and by u_i'; θ_i^v denotes the angle ($0 \leq \theta_i^v \leq \pi/2$) between the subspaces spanned by v_i and by v_i'; and[4]

$$\bar{\rho}_i \equiv \min\left\{2,\ \min_{j\neq i} \frac{|\sigma_j - \sigma_i|}{|\sigma_i|}\right\}$$

denotes the relative gap for the i^{th} singular value of B.

2. General perturbation results for the eigenproblem. This section presents general relative perturbation results for the eigenvalues and eigenvectors of a symmetric matrix.

The first theorem is the basis for all of the eigenvalue and singular value results to come. It contains the bound on the relative perturbation in the eigenvalues from which we derive a bound on the relative perturbation in the singular values in §3.

THEOREM 2.1. *Let $A + \delta A = D^T AD$, where D is a nonsingular matrix. Then*

$$\frac{|\lambda_i|}{\|(D^TD)^{-1}\|} \leq |\lambda_i'| \leq |\lambda_i|\, \|D^TD\|$$

and

$$|\lambda_i' - \lambda_i| \leq |\lambda_i|\, \|D^TD - I\|.$$

Proof. These inequalities are an immediate consequence of Ostrowski's Theorem [14, Theorem 4.5.9]; we include the proofs for completeness.

[3] Again we adopt the convention that $\rho_i = +\infty$ if $\lambda_i = 0$.

[4] We also adopt the convention that $\bar{\rho}_i = +\infty$ if $\sigma_i = 0$.

Since 0 is the i^{th} eigenvalue of $A - \lambda_i I$, Sylvester's Inertia Theorem [21] implies that 0 is the i^{th} eigenvalue of

$$D^T(A - \lambda_i I)D = (A + \delta A) - \lambda_i D^T D.$$

Using (1.1) with $M = A + \delta A$ and $\delta M = -\lambda_i D^T D$ gives

$$|0 - \lambda_i'| \leq \| - \lambda_i D^T D\|,$$

or

$$|\lambda_i'| \leq |\lambda_i| \, \|D^T D\|.$$

The derivation of the lower bound is similar. Furthermore,

$$D^T(A - \lambda_i I)D = (A + \delta A - \lambda_i I) + \lambda_i \, (I - D^T D).$$

Using (1.1) with $M = A + \delta A - \lambda_i I$ and $\delta M = \lambda_i \, (I - D^T D)$ yields

$$|0 - (\lambda_i' - \lambda_i)| \leq \|\lambda_i (I - D^T D)\|,$$

or

$$|\lambda_i' - \lambda_i| \leq |\lambda_i| \, \|D^T D - I\|.$$

□

These bounds are tight. For example, they give equality when D is a multiple of an orthogonal matrix.

The second theorem is the basis for all of the eigenvector and singular vector results to come. It bounds the angle between the spaces spanned by an exact eigenvector and the corresponding perturbed eigenvector in terms of the relative gap.

THEOREM 2.2. *Let $A + \delta A = D^T A D$, where D is a nonsingular matrix, and let*

$$\beta \equiv \|D - I\|, \qquad \gamma \equiv \|D^T D - I\|, \qquad \text{and} \qquad \delta \equiv \|D^T D\| \, \|D^{-T} D^{-1} - I\|.$$

Then

$$\sin \theta_i \leq \frac{\delta}{\rho_i - \gamma} + \beta,$$

provided that $\rho_i > \gamma$.

Proof. Let

$$M = A - \lambda_i' I, \quad \mu = \lambda_i - \lambda_i', \quad \delta M = \lambda_i'(I - D^{-T} D^{-1}), \quad \bar{\mu} = 0, \quad \text{and} \quad \bar{w} = D w_i',$$

so that

$$M w_i = \mu w_i \qquad \text{and} \qquad (M + \delta M)\bar{w} = 0\bar{w}.$$

Let $0 \leq \bar{\theta} \leq \pi/2$ be the angle between the spaces spanned by w_i and by $\bar{w}$. By (1.2),

$$\sin \bar{\theta} \leq |\lambda_i'| \, \|I - D^{-T} D^{-1}\| \, \left(\min_{j \neq i} |\lambda_j - \lambda_i'|\right)^{-1}$$

By Theorem 2.1,

$$|\lambda_i'| \leq |\lambda_i| \, \|D^T D\|$$

and

$$\min_{j \neq i} |\lambda_j - \lambda_i'| \geq \min_{j \neq i} |\lambda_j - \lambda_i| - |\lambda_i' - \lambda_i| \geq \min_{j \neq i} |\lambda_j - \lambda_i| - |\lambda_i| \, \|D^T D - I\|,$$

so that

$$\sin \bar{\theta} \leq |\lambda_i| \, \|D^T D\| \, \|I - D^{-T} D^{-1}\| \, \left(\min_{j \neq i} |\lambda_j - \lambda_i| - |\lambda_i| \, \|D^T D - I\|\right)^{-1} = \frac{\delta}{\rho_i - \gamma}.$$

To bound $\sin\theta_i$, we use the triangle inequality

$$\sin\theta_i \leq \sin\bar{\theta} + \sin\theta',$$

where $0 \leq \theta' \leq \pi/2$ is the angle between the spaces spanned by $\bar{w}$ and by w_i'. The second summand is bounded above by

$$\sin\theta' \leq \|\bar{w} - w_i'\| = \|(D-I)w_i'\| \leq \beta.$$

$\square$

3. General perturbation results for the singular value problem. This section presents general relative perturbation results for singular values and singular vectors. The results are derived by expressing the singular value decomposition as a symmetric eigenvalue decomposition. This can be done in two ways. First, the positive square-roots of the eigenvalues of B^TB or BB^T are the singular values of B; i.e., $\sigma_i = \sqrt{\lambda_i[B^TB]} = \sqrt{\lambda_i[BB^T]}$, $1 \leq i \leq \min\{m,n\}$. This transformation is used in Theorem 3.1 to derive bounds on the relative perturbation in the singular values. Second, the eigenvalues of the Jordan-Wielandt matrix

$$A = \begin{pmatrix} 0 & B^T \\ B & 0 \end{pmatrix}$$

are $\pm\sigma_i$ (and 0 if $m \neq n$), and

$$w_i = \frac{1}{\sqrt{2}}\begin{pmatrix} v_i \\ u_i \end{pmatrix}$$

is a unit eigenvector of A associated with $\lambda_i = \sigma_i$ (see [10], [22, Theorem I.4.2]). This transformation is used in Theorem 3.3 to derive the result for the singular vectors.

THEOREM 3.1. *Let $B + \delta B = D_L B D_R$, where D_L and D_R are nonsingular matrices. Then*

$$\frac{\sigma_i}{\|D_L^{-1}\|\,\|D_R^{-1}\|} \leq \sigma_i' \leq \sigma_i\,\|D_L\|\,\|D_R\|.$$

Proof. Let $C = D_L B$. Apply Theorem 2.1 with $A = C^TC$ and $D = D_R$ to get

$$\frac{\lambda_i[C^TC]}{\|D_R^{-1}\|^2} \leq \lambda_i[D_R^T(C^TC)D_R] = \sigma_i'^2 \leq \lambda_i[C^TC]\,\|D_R\|^2;$$

and again with $A = BB^T$ and $D = D_L^T$ to get

$$\frac{\lambda_i[BB^T]}{\|D_L^{-1}\|^2} \leq \lambda_i[D_L(BB^T)D_L^T] = \lambda_i[CC^T] \leq \lambda_i[BB^T]\,\|D_L\|^2.$$

But $\sigma_i^2 = \lambda_i[BB^T]$ and $\lambda_i[C^TC] = \lambda_i[CC^T]$, for $1 \leq i \leq \min\{m,n\}$. $\square$

This result could also be proved by means of the inequalities

$$\sigma_n[D]\sigma_i[B] \leq \sigma_j[DB] \leq \sigma_1[D]\sigma_i[B],$$

which are easily derived from the case $j = 1$ of [15, Theorem 3.3.16].

COROLLARY 3.2 (BARLOW AND DEMMEL [1, p. 771]). *Let $B + \delta B = D_L B D_R$, where D_L and D_R are nonsingular diagonal matrices. Then*

$$\sigma_i \cdot \min_j |(D_L)_{jj}| \cdot \min_k |(D_R)_{kk}| \;\leq\; \sigma_i' \;\leq\; \sigma_i \cdot \max_j |(D_L)_{jj}| \cdot \max_k |(D_R)_{kk}|.$$

THEOREM 3.3. *Let $B + \delta B = D_L B D_R$, where D_L and D_R are nonsingular matrices, and let*

$$\bar\beta \equiv \max\{\|D_L^T - I\|, \|D_R - I\|\}, \qquad \bar\gamma \equiv \max\{\|D_L D_L^T - I\|, \|D_R^T D_R - I\|\},$$

and

$$\bar\delta \equiv \max\{\|D_L D_L^T\|, \|D_R^T D_R\|\} \cdot \max\{\|D_L^{-1} D_L^{-T} - I\|, \|D_R^{-T} D_R^{-1} - I\|\}.$$

Then

$$|\sigma_i' - \sigma_i| \le \bar\gamma\, \sigma_i;$$

and, provided that $\bar\rho_i > \bar\gamma$,

$$\max\{\sin\theta_i^u, \sin\theta_i^v\} \le \sqrt{2}\left(\frac{\bar\delta}{\bar\rho_i - \bar\gamma} + \bar\beta\right).$$

Proof. Let

$$A = \begin{pmatrix} 0 & B^T \\ B & 0 \end{pmatrix}, \quad A + \delta A = \begin{pmatrix} 0 & (B+\delta B)^T \\ B + \delta B & 0 \end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix} D_R & \\ & D_L^T \end{pmatrix},$$

so that

$$w_i = \frac{1}{\sqrt{2}}\begin{pmatrix} v_i \\ u_i \end{pmatrix}, \qquad w_i' = \frac{1}{\sqrt{2}}\begin{pmatrix} v_i' \\ u_i' \end{pmatrix}, \qquad \text{and} \qquad A + \delta A = D^T A D.$$

The bound for the singular values follows directly from Theorem 2.1. By Theorem 2.2,

$$\sin\theta_i \le \frac{\delta}{\rho_i - \gamma} + \beta,$$

where

$$\beta = \|D - I\|, \qquad \gamma = \|D^T D - I\|, \qquad \delta = \|D^T D\|\, \|D^{-T} D^{-1} - I\|,$$

and

$$\rho_i = \min_{\ell \ne i} \frac{|\lambda_\ell - \sigma_i|}{\sigma_i},$$

provided that $\rho_i > \gamma$. But $\beta = \bar\beta$, $\gamma = \bar\gamma$, $\delta = \bar\delta$, and $\rho_i = \bar\rho_i$ (recall that $\lambda_\ell = \pm\sigma_j$ for some j), so that

$$\sin\theta_i \le \frac{\bar\delta}{\bar\rho_i - \bar\gamma} + \bar\beta,$$

provided that $\bar\rho_i > \bar\gamma$. Now

$$\sin\theta_i^u = \min_\alpha \|u_i' - \alpha u_i\| \le \|u_i' - (\cos\theta) u_i\| \le \sqrt{2}\, \|w_i' - (\cos\theta) w_i\| = \sqrt{2}\, \sin\theta_i.$$

The derivation of the bound for $\sin\theta_i^v$ is similar. □

In the remaining sections we apply the general perturbation results in this section and the previous one to particular classes of matrices and perturbations.

4. Componentwise relative perturbations. This section deals with perturbations that cause relative changes in the nonzero elements of tridiagonal matrices with zero diagonal elements, bidiagonal matrices, and biacyclic matrices. By applying the results in §2 and §3, we get simple proofs of many of the relative perturbation results in [1, 5, 8, 6] and solve an open problem in [8].

4.1. Eigenvalues and singular values. Symmetric tridiagonal matrices with zero diagonal elements are symmetric permutations of matrices of the form

$$\begin{pmatrix} 0 & B^T \\ B & 0 \end{pmatrix},$$

where B is bidiagonal. They result from formulating the singular value problem as a symmetric eigenvalue problem (see §3). We begin by bounding the effect of a relative change in a single pair of nonzero elements.

COROLLARY 4.1 (KAHAN [18], DEMMEL AND KAHAN [6, THEOREM 2]). *Let A be a symmetric tridiagonal matrix with zero diagonal elements, and let $A + \delta A$ equal A except for the off-diagonal elements $a_{k,k+1} = a_{k+1,k}$ that are perturbed to $\alpha a_{k,k+1} = \alpha a_{k+1,k}$, for some $\alpha \neq 0$. Then*

$$\frac{1}{1+\eta}\,|\lambda_i| \leq |\lambda_i'| \leq (1+\eta)\,|\lambda_i|,$$

where $\eta \equiv \max\{|\alpha|, 1/|\alpha|\} - 1$.

Proof. Let

$$\begin{array}{rccccccccc} & & & & k & k+1 & & & \\ D = \mathrm{diag} & \big(\cdots & \sqrt{\alpha} & 1/\sqrt{\alpha} & \sqrt{\alpha} & \sqrt{\alpha} & 1/\sqrt{\alpha} & \sqrt{\alpha} & \cdots \big), \end{array}$$

so that $A + \delta A = D^T A D$ (see [1, p. 770]). Now apply Theorem 2.1, and note that $\|D^T D\| = \|(D^T D)^{-1}\| = 1 + \eta$. $\square$

By repeated application of this result we can bound the effect of relative changes in every pair of nonzero elements (see [6, Corollary 1]). The result is analogous to the following result for bidiagonal matrices.

COROLLARY 4.2 (BARLOW AND DEMMEL [1, THEOREM 1], DEIFT ET AL. [5, THEOREM 2.12], DEMMEL AND KAHAN [6, COROLLARY 2]). *Let*

$$B = \begin{pmatrix} b_{11} & b_{12} & & \\ & b_{22} & \ddots & \\ & & \ddots & b_{n-1,n} \\ & & & b_{nn} \end{pmatrix} \quad \text{and} \quad B + \delta B = \begin{pmatrix} \alpha_1 b_{11} & \alpha_2 b_{12} & & \\ & \alpha_3 b_{22} & \ddots & \\ & & \ddots & \alpha_{2n-2} b_{n-1,n} \\ & & & \alpha_{2n-1} b_{nn} \end{pmatrix},$$

where $\alpha_j \neq 0$. Then

$$\frac{1}{1+\eta}\,\sigma_i \leq \sigma_i' \leq (1+\eta)\,\sigma_i,$$

where $\eta \equiv \prod_{j=1}^{2n-1} \max\{|\alpha_j|, 1/|\alpha_j|\} - 1$.

Proof. Let

$$D_L = \mathrm{diag}\left(\alpha_1, \frac{\alpha_1\alpha_3}{\alpha_2}, \frac{\alpha_1\alpha_3\alpha_5}{\alpha_2\alpha_4}, \frac{\alpha_1\alpha_3\alpha_5\alpha_7}{\alpha_2\alpha_4\alpha_6}, \cdots\right)$$

and

$$D_R = \mathrm{diag}\left(1, \frac{\alpha_2}{\alpha_1}, \frac{\alpha_2\alpha_4}{\alpha_1\alpha_3}, \frac{\alpha_2\alpha_4\alpha_6}{\alpha_1\alpha_3\alpha_5}, \cdots\right),$$

so that $B + \delta B = D_L B D_R$ (see [1, p. 770]). Now apply Theorem 3.1 and note that $\|D_L\|\,\|D_R\| \leq 1 + \eta$ and $\|D_L^{-1}\|\,\|D_R^{-1}\| \leq 1 + \eta$. $\square$

Biacyclic matrices are a generalization of bidiagonal matrices in which the underlying bipartite graph can be any acyclic bipartite graph [8]. Only biacyclic matrices have the property that relative perturbations of the matrix elements cause relative perturbations in the singular values that are independent of the values of the elements [8].

COROLLARY 4.3 (DEMMEL AND GRAGG [8, THEOREM 1]). *Let B be biacyclic, and let $B+\delta B$ be a componentwise relative perturbation of B; i.e., $[B+\delta B]_{k,\ell} = \alpha_{k,\ell} b_{k,\ell}$ for all k and ℓ, where $\alpha_{k,\ell} \neq 0$. Then*

$$\frac{1}{1+\eta}\,\sigma_i \leq \sigma_i' \leq (1+\eta)\,\sigma_i,$$

where $\eta \equiv \prod_{b_{k,\ell}\neq 0} \max\{|\alpha_{k,\ell}|, 1/|\alpha_{k,\ell}|\} - 1$.

Proof. By Lemma 1 in [8], there exist diagonal matrices D_L and D_R such that $B+\delta B = D_L B D_R$. The diagonal entries of D_L and D_R are quotients of products of distinct $\alpha_{k,\ell}$, where each $\alpha_{k,\ell}$ can only appear either in the numerators of D_L and the denominators of D_R, or in the numerators of D_R and the denominators of D_L. Now apply Theorem 3.1, and verify that $\|D_L\|\,\|D_R\| \leq 1+\eta$ and $\|D_L^{-1}\|\,\|D_R^{-1}\| \leq 1+\eta$. □

4.2. Eigenvectors and singular vectors. The corresponding results for eigenvectors and singular vectors again follow directly from the results in §2 and §3.

COROLLARY 4.4. *Let A be a symmetric tridiagonal matrix with zero diagonal elements, and let $A+\delta A$ equal A except for the off-diagonal elements $a_{k,k+1} = a_{k+1,k}$ that change to $\alpha a_{k,k+1} = \alpha a_{k+1,k}$, for some $\alpha > 0$. Then*

$$\sin\theta_i \leq \frac{\eta(1+\eta)}{\rho_i - \eta} + \frac{\eta}{2},$$

where $\eta \equiv \max\{\alpha, 1/\alpha\} - 1$, provided that $\rho_i > \eta$.

Proof. Define D as in the proof of Corollary 4.1 so that $A+\delta A = D^T A D$. Now apply Theorem 2.2, and note that

$$\beta = \max\{|\sqrt{\alpha}-1|, |1/\sqrt{\alpha}-1|\} \leq \frac{\eta}{2},$$

$$\gamma = \max\{|\alpha-1|, |1/\alpha - 1|\} = \eta,$$

and

$$\delta = \max\{\alpha, 1/\alpha\}\cdot\max\{|\alpha-1|, |1/\alpha-1|\} \leq (1+\eta)\,\eta.$$

□

Deift et al. [5, p. 1472] prove that

$$\sin\theta_i \leq \frac{\eta(1+\eta')}{\mathrm{relgap}_i - \eta'},$$

where

$$\eta' \equiv e^{\eta} - 1 \qquad \text{and} \qquad \mathrm{relgap}_i \equiv \min_{j\neq i} \frac{|\lambda_j - \lambda_i|}{|\lambda_j| + |\lambda_i|},$$

provided that $\mathrm{relgap}_i > \eta'$. If ρ_i is small, then $\mathrm{relgap}_i \approx \rho_i/2$. Hence, if η is also small, then this bound is slightly weaker than the bound in Corollary 4.4.

COROLLARY 4.5. *Let*

$$B = \begin{pmatrix} b_{11} & b_{12} & & \\ & b_{22} & \ddots & \\ & & \ddots & b_{n-1,n} \\ & & & b_{nn} \end{pmatrix} \quad \textit{and} \quad B+\delta B = \begin{pmatrix} \alpha_1 b_{11} & \alpha_2 b_{12} & & \\ & \alpha_3 b_{22} & \ddots & \\ & & \ddots & \alpha_{2n-2} b_{n-1,n} \\ & & & \alpha_{2n-1} b_{nn} \end{pmatrix},$$

where $\alpha_j > 0$. *Then*

$$\max\{\sin\theta_i^u, \sin\theta_i^v\} \le \sqrt{2}\left(\frac{\eta(1+\eta)}{\bar\rho_i - \eta} + \frac{\eta}{2}\right),$$

where $\eta \equiv \prod_{j=1}^{2n-1} \max\{\alpha_j, 1/\alpha_j\} - 1$, *provided that* $\bar\rho_i > \eta$.

Proof. Let

$$D_L = \frac{1}{\alpha_0}\operatorname{diag}\left(\alpha_1, \frac{\alpha_1\alpha_3}{\alpha_2}, \frac{\alpha_1\alpha_3\alpha_5}{\alpha_2\alpha_4}, \frac{\alpha_1\alpha_3\alpha_5\alpha_7}{\alpha_2\alpha_4\alpha_6}, \cdots\right)$$

and

$$D_R = \alpha_0 \operatorname{diag}\left(1, \frac{\alpha_2}{\alpha_1}, \frac{\alpha_2\alpha_4}{\alpha_1\alpha_3}, \frac{\alpha_2\alpha_4\alpha_6}{\alpha_1\alpha_3\alpha_5}, \cdots\right),$$

so that $B + \delta B = D_L B D_R$ (see [1, p. 770]). Choose α_0 to minimize

$$\max_i \{(D_L)_{jj}, (D_R)_{jj}, (D_L^{-1})_{jj}, (D_R^{-1})_{jj}\}$$

so that

$$\frac{1}{\sqrt{1+\eta}} \le (D_L)_{jj},\ (D_R)_{jj},\ (D_L^{-1})_{jj},\ (D_R^{-1})_{jj} \le \sqrt{1+\eta}.$$

Now apply Theorem 3.3 and note that $\bar\beta \le \frac{\eta}{2}$, $\bar\gamma \le \eta$, and $\bar\delta \le \eta(1+\eta)$ (cf. the proof of Corollary 4.4). □

Deift et al. [5, Theorem 2.12] prove that

$$\max\{\sin\theta_i^u, \sin\theta_i^v\} \le \sqrt{2}\frac{\eta'(1+\eta')}{\mathrm{relgap}_i - \eta'},$$

where

$$\eta' \equiv e^\eta - 1 \qquad \text{and} \qquad \mathrm{relgap}_i \equiv \min_{j\ne i} \frac{|\sigma_j - \sigma_i|}{\sigma_j + \sigma_i},$$

provided that $\mathrm{relgap}_i > \eta'$. Again, when $\bar\rho_i$ and η are small, this bound is slightly weaker than the bound in Corollary 4.5.

The following result solves an open problem in [8]; the proof is similar to the proof of Corollary 4.5.

COROLLARY 4.6. *Let* B *be biacyclic, and let* $B+\delta B$ *be a componentwise relative perturbation of* B*; i.e.,* $[B+\delta B]_{k,\ell} = \alpha_{k,\ell} b_{k,\ell}$ *for all* k *and* ℓ*, where* $\alpha_{k,\ell} > 0$. *Then*

$$\max\{\sin\theta_i^u, \sin\theta_i^v\} \le \sqrt{2}\left(\frac{\eta(1+\eta)}{\bar\rho_i - \eta} + \frac{\eta}{2}\right),$$

where $\eta \equiv \prod_{b_{k,\ell}\ne 0} \max\{\alpha_{k,\ell}, 1/\alpha_{k,\ell}\} - 1$, *provided that* $\bar\rho_i > \eta$.

5. Perturbations that change the sparsity pattern. This section presents bounds on the relative perturbation in the singular values and singular vectors of a block triangular or bidiagonal matrix when an off-diagonal block is changed. This change is called *deflation* if the off-diagonal block is set to zero. By applying the results in §2 and §3, we get simple proofs of many of the deflation results in [2, 5, 6, 16, 20] and strengthen the deflation result in [11].

5.1. Block triangular matrices. First we derive bounds on the relative perturbation in the singular values and vectors of block triangular matrices. Then we derive bounds on the singular values of the diagonal blocks.

LEMMA 5.1. *Let $B + \delta B = D_L B$ or $B + \delta B = B D_R$, where*

$$D_L = \begin{array}{c c} & \begin{array}{cc} k & m-k \end{array} \\ \begin{array}{c} k \\ m-k \end{array} & \begin{pmatrix} I & X \\ & I \end{pmatrix} \end{array} \qquad \text{or} \qquad D_R = \begin{array}{c c} & \begin{array}{cc} k & n-k \end{array} \\ \begin{array}{c} k \\ n-k \end{array} & \begin{pmatrix} I & X \\ & I \end{pmatrix} \end{array}.$$

Then

$$\frac{1}{1+\eta}\,\sigma_i \le \sigma_i' \le (1+\eta)\,\sigma_i$$

and

$$|\sigma_i' - \sigma_i| \le \eta\,\sigma_i,$$

where $\eta \equiv \|X\|$. Moreover,

$$\max\{\sin\theta_i^u, \sin\theta_i^v\} \le \sqrt{2}\left(\frac{\eta'(1+\eta')}{\bar\rho_i - \eta'} + \frac{\eta}{2}\right),$$

where $\eta' \equiv e^{\eta} - 1$, provided that $\bar\rho_i > \eta'$.

Proof. We treat the case $B + \delta B = D_L B$; the other case is similar. Since

$$D_L^{-1} = \begin{pmatrix} I & -X \\ & I \end{pmatrix}$$

we have $\|D_L\| = \|D_L^{-1}\| \le 1+\eta$. To get the first pair of inequalities, apply Theorem 3.1 with $D_R = I$. The second inequality follows immediately. The final inequality follows from Theorem 3.3 after noting that

$$\bar\beta = \left\|\begin{pmatrix} 0 & 0 \\ X^T & 0 \end{pmatrix}\right\| = \eta, \qquad \bar\gamma = \left\|\begin{pmatrix} XX^T & X \\ X^T & 0 \end{pmatrix}\right\| \le \eta',$$

and

$$\bar\delta = \left\|\begin{pmatrix} I + XX^T & X \\ X^T & I \end{pmatrix}\right\|\left\|\begin{pmatrix} XX^T & -X \\ -X^T & 0 \end{pmatrix}\right\| \le (1+\eta')\,\eta',$$

where we have used the fact that

$$\left\|\begin{pmatrix} \alpha^2 & \alpha \\ \alpha & 0 \end{pmatrix}\right\| \le e^{\alpha} - 1$$

when $\alpha \ge 0$. ☐

When B is block triangular, we can bound the effect of setting the off-diagonal block to zero.

THEOREM 5.2. *Let*

$$B = \begin{array}{c c} & \begin{array}{cc} k & n-k \end{array} \\ \begin{array}{c} k \\ n-k \end{array} & \begin{pmatrix} B_{11} & B_{12} \\ & B_{22} \end{pmatrix} \end{array} \qquad \text{and} \qquad B + \delta B = \begin{pmatrix} B_{11} & \\ & B_{22} \end{pmatrix},$$

where B_{11} or B_{22} is nonsingular. Then

$$|\sigma_i' - \sigma_i| \le \sigma_i\,\eta,$$

where

$$\eta \equiv \frac{\|B_{12}\|}{\max\{\sigma_{min}[B_{11}], \sigma_{min}[B_{22}]\}}.$$

Moreover,

$$\max\{\sin\theta_i^u, \sin\theta_i^v\} \leq \sqrt{2}\left(\frac{\eta'(1+\eta')}{\bar{\rho}_i - \eta'} + \frac{\eta}{2}\right),$$

where $\eta' \equiv e^\eta - 1$, *provided that* $\bar{\rho}_i > \eta'$. *Finally, if* $\sigma_{min}[B_{11}] > \sigma_{max}[B_{22}]$, *then*

$$0 \leq \sigma_i - \sigma_i[B_{11}] \leq \sigma_i \, \frac{\|B_{12}\|}{\sigma_{min}[B_{11}]}, \qquad 1 \leq i \leq k$$

and

$$0 \leq \sigma_i[B_{22}] - \sigma_{k+i} \leq \sigma_{k+i} \, \frac{\|B_{12}\|}{\sigma_{min}[B_{11}]}, \qquad 1 \leq i \leq n-k.$$

Proof. If B_{11} is nonsingular, then

$$\begin{pmatrix} B_{11} & B_{12} \\ & B_{22} \end{pmatrix} \begin{pmatrix} I & -B_{11}^{-1}B_{12} \\ & I \end{pmatrix} = \begin{pmatrix} B_{11} & \\ & B_{22} \end{pmatrix},$$

and by Lemma 5.1,

$$|\sigma_i' - \sigma_i| \leq \sigma_i \, \|B_{11}^{-1}B_{12}\| \leq \sigma_i \, \frac{\|B_{12}\|}{\sigma_{min}[B_{11}]}.$$

If B_{22} is nonsingular, then

$$\begin{pmatrix} I & -B_{12}B_{22}^{-1} \\ & I \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ & B_{22} \end{pmatrix} = \begin{pmatrix} B_{11} & \\ & B_{22} \end{pmatrix},$$

and by Lemma 5.1,

$$|\sigma_i' - \sigma_i| \leq \sigma_i \, \|B_{12}B_{22}^{-1}\| \leq \sigma_i \, \frac{\|B_{12}\|}{\sigma_{min}[B_{22}]}.$$

This proves the first inequality. The second inequality follows from Lemma 5.1.

If $\sigma_{min}[B_{11}] > \sigma_{max}[B_{22}]$, then

$$\begin{aligned} \sigma_i' &= \sigma_i[B_{11}], \qquad 1 \leq i \leq k, \\ \sigma_{k+i}' &= \sigma_i[B_{22}], \qquad 1 \leq i \leq n-k, \end{aligned}$$

and

$$\max\{\sigma_{min}[B_{11}], \sigma_{min}[B_{22}]\} = \sigma_{min}[B_{11}].$$

The lower bounds in the last two sets of inequalities follow from the Interlacing Theorem for singular values [13, Corollary 8.3.3]. □

Weaker bounds on the singular values appear in [2].

COROLLARY 5.3 (CHANDRASEKARAN AND IPSEN [2, THEOREM 5.2.1]). *With the notation of Theorem 5.2, assume that* $\sigma_{min}[B_{11}] > \sigma_{max}[B_{22}]$. *Then*

$$|\sigma_i[B_{22}] - \sigma_{k+i}| \leq \sigma_{max}[B_{22}] \frac{\|B_{12}\|}{\sigma_{min}[B_{11}] - \sigma_{max}[B_{22}]}, \qquad 1 \leq i \leq n-k.$$

Proof. By Theorem 5.2,

$$0 \leq \sigma_i[B_{22}] - \sigma_{k+i} \leq \sigma_{k+i} \, \frac{\|B_{12}\|}{\sigma_{min}[B_{11}]}.$$

Thus

$$\sigma_{k+i} \leq \sigma_i[B_{22}] \leq \sigma_{max}[B_{22}],$$

and the result follows immediately. □

By applying Theorem 3.1 directly, we get the following result.

COROLLARY 5.4. *With the notation of Theorem 5.2, assume that* $\sigma_{min}[B_{11}] > \sigma_{max}[B_{22}]$. *Then*

$$\sigma_i[B_{11}] \le \sigma_i \le \left(1 + \frac{\|B_{12}\|^2}{\sigma_{min}[B_{11}]^2 - \sigma_{max}[B_{22}]^2}\right)^{1/2} \sigma_i[B_{11}], \qquad 1 \le i \le k$$

and

$$\left(1 + \frac{\|B_{12}\|^2}{\sigma_{min}[B_{11}]^2 - \sigma_{max}[B_{22}]^2}\right)^{-1/2} \sigma_i[B_{22}] \le \sigma_{k+i} \le \sigma_i[B_{22}], \qquad 1 \le i \le n-k.$$

Proof. By Theorem 5.2, we have $\sigma_i[B_{11}] \le \sigma_i$. Let $\rho \equiv \sigma_{max}[B_{22}]/\sigma_{min}[B_{11}] < 1$. Since B_{11} is nonsingular,

$$B \equiv \begin{pmatrix} B_{11} & B_{12} \\ & B_{22} \end{pmatrix} = \begin{pmatrix} B_{11} & \\ & \frac{1}{\rho} B_{22} \end{pmatrix} \begin{pmatrix} I & B_{11}^{-1} B_{12} \\ & \rho I \end{pmatrix} \equiv \hat{B}\hat{D}.$$

By Theorem 3.1 with $D_L = I$, and $D_R = \hat{D}$, we have $\sigma_i \le \sigma_i[\hat{B}]\, \|\hat{D}\|$. Since $\sigma_{max}[\frac{1}{\rho}\, B_{22}] = \sigma_{min}[B_{11}]$,

$$\sigma_i[\hat{B}] = \sigma_i[B_{11}], \qquad 1 \le i \le k.$$

Now

$$\begin{aligned} \|\hat{D}\| &= \max_j \left\| \begin{pmatrix} 1 & \sigma_j[B_{11}^{-1} B_{12}] \\ & \rho \end{pmatrix} \right\| \le \max_j \left(1 + \frac{\sigma_j[B_{11}^{-1} B_{12}]^2}{1-\rho^2}\right)^{1/2} \\ &= \left(1 + \frac{\|B_{11}^{-1} B_{12}\|^2}{1-\rho^2}\right)^{1/2} \le \left(1 + \frac{\|B_{12}\|^2}{\sigma_{min}[B_{11}]^2 - \sigma_{max}[B_{22}]^2}\right)^{1/2}. \end{aligned}$$

The second inequality follows in a similar fashion from the identity

$$B\hat{D} \equiv \begin{pmatrix} B_{11} & B_{12} \\ & B_{22} \end{pmatrix} \begin{pmatrix} \rho I & -B_{11}^{-1} B_{12} \\ & I \end{pmatrix} = \begin{pmatrix} \rho B_{11} & \\ & B_{22} \end{pmatrix} \equiv \hat{B}.$$

□

Mathias and Stewart [20, Theorem 3.1] prove the slightly weaker results

$$\sigma_i[B_{11}] \le \sigma_i \le \left(1 - \frac{\|B_{12}\|^2}{\sigma_{min}[B_{11}]^2 - \sigma_{max}[B_{22}]^2}\right)^{-1/2} \sigma_i[B_{11}], \qquad 1 \le i \le k$$

and

$$\left(1 - \frac{\|B_{12}\|^2}{\sigma_{min}[B_{11}]^2 - \sigma_{max}[B_{22}]^2}\right)^{1/2} \sigma_i[B_{22}] \le \sigma_{k+i} \le \sigma_i[B_{22}], \qquad 1 \le i \le n-k$$

that are used in their analysis of the URV decomposition. For the special case where B is bidiagonal and $k = n-1$, Demmel and Kahan [6, Theorem 5] prove the slightly weaker result

$$|\sigma_i' - \sigma_i| \le 2\, \sigma_i' \,\frac{\|B_{12}\|^2}{\sigma_{min}[B_{11}]^2 - \sigma_{max}[B_{22}]^2}.$$

5.2. Bidiagonal matrices. This section applies the results of §5.1 to bidiagonal matrices. In particular, we derive the deflation criteria used in [5, 6] to split a matrix into independent submatrices.

THEOREM 5.5. *Let*

$$B = \begin{array}{c c} & \begin{array}{cc} k & n-k \end{array} \\ \begin{array}{c} k \\ n-k \end{array} & \begin{pmatrix} B_{11} & b_{k,k+1} e_k e_1^T \\ & B_{22} \end{pmatrix} \end{array} \qquad \text{and} \qquad B + \delta B = \begin{pmatrix} B_{11} & \\ & B_{22} \end{pmatrix}$$

be nonsingular bidiagonal matrices. Then

$$\frac{\sigma_i}{1+\eta} \le \sigma_i' \le (1+\eta)\,\sigma_i,$$

where

$$\eta \equiv |b_{k,k+1}| \cdot \min\{\|B_{11}^{-1} e_k\|,\ \|B_{22}^{-T} e_1\|\}.$$

Moreover,

$$\max\{\sin\theta_i^u, \sin\theta_i^v\} \le \sqrt{2}\left(\frac{\eta'(1+\eta')}{\bar\rho_i - \eta'} + \frac{\eta}{2}\right),$$

where $\eta' \equiv e^{\eta} - 1$, provided that $\bar\rho_i > \eta'$.

Proof. As in the proof of Theorem 5.2, we apply Lemma 5.1 with $B + \delta B = BD_R$ and $X = -b_{k,k+1} B_{11}^{-1} e_k e_1^T$, so that

$$\|X\| \le |b_{k,k+1}|\ \|B_{11}^{-1} e_k\|;$$

or with $B + \delta B = D_L B$ and $X = -b_{k,k+1} e_k e_1^T B_{22}^{-1}$, so that

$$\|X\| \le |b_{k,k+1}|\ \|B_{22}^{-T} e_1\|.$$

The result follows immediately. □

Now we show that the above theorem justifies Convergence Criterion 1 from [5, 6], which is used in the zero-shift Golub-Kahan algorithm for computing the singular values of bidiagonal matrices to high relative accuracy.

COROLLARY 5.6. *Assume that*

$$B = \begin{pmatrix} b_{11} & b_{12} & & \\ & b_{22} & \ddots & \\ & & \ddots & b_{n-1,n} \\ & & & b_{nn} \end{pmatrix}$$

is nonsingular, and let $B + \delta B$ equal B except that the $(k, k+1)$ entry is set to zero. Let

$$\eta \equiv |b_{k,k+1}| \cdot \min\left\{\|S^{-1} e_{2k}\|,\ \|S^{-1} e_{2k+1}\|\right\},$$

where

$$S = \begin{pmatrix} 0 & b_{11} & & & & \\ b_{11} & 0 & b_{12} & & & \\ & b_{12} & 0 & \ddots & & \\ & & \ddots & \ddots & b_{n-1,n} & \\ & & & b_{n-1,n} & 0 & b_{nn} \\ & & & & b_{nn} & 0 \end{pmatrix},$$

and let $\eta' \equiv e^{\eta} - 1$. Then

$$\frac{\sigma_i}{1+\eta} \le \sigma_i' \le (1+\eta)\,\sigma_i.$$

Moreover,

$$\max\{\sin\theta_i^u, \sin\theta_i^v\} \le \sqrt{2}\left(\frac{\eta'(1+\eta')}{\bar\rho_i - \eta'} + \frac{\eta}{2}\right),$$

provided that $\bar\rho_i > \eta'$. Finally, these results hold when η and η' are replaced by

$$\eta_1 \equiv |b_{k,k+1}| \cdot \min\left\{\|S^{-1}e_{2k}\|_1,\ \|S^{-1}e_{2k+1}\|_1\right\} \qquad \text{and} \qquad \eta_1' \equiv e^{\eta_1} - 1.$$

Proof. Let

$$B = \begin{array}{c c} & \begin{array}{cc} k & n-k \end{array} \\ \begin{array}{c} k \\ n-k \end{array} & \begin{pmatrix} B_{11} & B_{12} \\ & B_{22} \end{pmatrix} \end{array}.$$

From Theorem 5.5, it suffices to show that

$$\|B_{11}^{-1}e_k\| = \|S^{-1}e_{2k}\| \qquad \text{and} \qquad \|B_{22}^{-T}e_1\| = \|S^{-1}e_{2k+1}\|.$$

But

$$S = P\begin{pmatrix} & B^T \\ B & \end{pmatrix}P^T,$$

where the permutation matrix P represents the mapping

$$P: (1\ \ 2\ \ 3\ \ 4\ \ \cdots\ \ 2n-1\ \ 2n) \mapsto (1\ \ n+1\ \ 2\ \ n+2\ \ \cdots\ \ n\ \ 2n).$$

Hence

$$S^{-1}e_{2k} = P\begin{pmatrix} & B^{-1} \\ B^{-T} & \end{pmatrix}P^T e_{2k} = P\begin{pmatrix} B^{-1}e_k \\ 0_n \end{pmatrix} = P\begin{pmatrix} B_{11}^{-1}e_k \\ 0_{2n-k} \end{pmatrix},$$

where 0_j represents a $j \times 1$ zero vector, and

$$S^{-1}e_{2k+1} = P\begin{pmatrix} & B^{-1} \\ B^{-T} & \end{pmatrix}P^T e_{2k+1} = P\begin{pmatrix} 0_n \\ B^{-T}e_{k+1} \end{pmatrix} = P\begin{pmatrix} 0_{n+k} \\ B_{22}^{-T}e_1 \end{pmatrix}.$$

The final result follows from the vector-norm inequality $\|\cdot\| \le \|\cdot\|_1$. □

Demmel and Kahan [6, Theorem 4] prove that

$$e^{-\eta/\sqrt{2}}\sigma_i \le \sigma_i' \le e^{\eta/\sqrt{2}}\,\sigma_i,$$

which is slightly stronger than the bound in Corollary 5.6 when η is small.[5] Deift et al. [5, Theorem 4.7] prove that

$$\frac{\sigma_i}{1+\eta'} \le \sigma_i' \le (1+\eta')\,\sigma_i,$$

which is slightly weaker.

Deift et al. [5, Theorem 4.7] also prove that

$$\max\{\sin\theta_i^u, \sin\theta_i^v\} \le \frac{\eta'(1+\eta')}{\mathrm{relgap}_i - \eta'} + \eta',$$

and

$$\max\{\sin\theta_i^u, \sin\theta_i^v\} \le \sqrt{\frac{2n+5}{4}}\,\frac{\eta_1'(1+\eta_1')}{\mathrm{relgap}_i - \eta_1'} + \frac{1}{\sqrt{2}}\left(1 + \sqrt{n - \frac{1}{2}}\right)\eta_1',$$

where

$$\mathrm{relgap}_i \equiv \min_{j \ne i} \frac{|\sigma_j - \sigma_i|}{\sigma_j + \sigma_i}.$$

If $\bar\rho_i$ and η are small, then $\mathrm{relgap}_i \approx \bar\rho_i/2$ and $\eta' \approx \eta$, so that these bounds are slightly weaker than the bounds in Corollary 5.6.

[5] This bound can be obtained by sharpening the bound on $\|D_L\|$ or $\|D_R\|$ in Lemma 5.1 to $e^{\eta/\sqrt{2}}$.

5.3. Shifted matrices. Faddeev, Kublanovskaya, and Faddeeva [9] and Chandrasekaran and Ipsen [2, 16] present an unshifted QR algorithm for computing the singular values and vectors of a triangular matrix B. The algorithm is a generalization of the Golub-Kahan algorithm [12] for bidiagonal matrices because it does not form B^TB or BB^T. All of the deflation criteria presented in §5.1 can be incorporated into this algorithm. The following result presents a deflation criterion for a shifted version.

We follow the approach in [11] for bidiagonal matrices and incorporate the shift τ by introducing the matrix $(\tilde{B} \quad \tau I)$, where $BB^T = \tilde{B}\tilde{B}^T + \tau^2 I$. Note that B and $(\tilde{B} \quad \tau I)$ have the same singular values and left singular vectors.

THEOREM 5.7. *Let*

$$\tilde{B} = \begin{array}{c} \\ k \\ n-k \end{array}\!\!\begin{array}{c} \begin{array}{cc} k & n-k \end{array} \\ \begin{pmatrix} \tilde{B}_{11} & \tilde{B}_{12} \\ & \tilde{B}_{22} \end{pmatrix} \end{array} \qquad \text{and} \qquad \tilde{B} + \delta\tilde{B} = \begin{pmatrix} \tilde{B}_{11} & \\ & \tilde{B}_{22} \end{pmatrix},$$

and let τ be a real number such that $(\tilde{B}_{11} \quad \tau I)$ has full rank. Assume that

$$BB^T = \tilde{B}\tilde{B}^T + \tau^2 I \quad \text{and} \quad (B + \delta B)(B + \delta B)^T = (\tilde{B} + \delta\tilde{B})(\tilde{B} + \delta\tilde{B})^T + \tau^2 I.$$

Then

$$|\sigma_i' - \sigma_i| \leq \sigma_i\, \eta,$$

where

$$\eta \equiv \frac{\|\tilde{B}_{12}\|}{\sqrt{\sigma_{min}[\tilde{B}_{11}]^2 + \tau^2}}.$$

Moreover,

$$\sin\theta_i^u \leq \sqrt{2}\left(\frac{\eta'(1+\eta')}{\bar{\rho}_i - \eta'} + \frac{\eta}{2}\right),$$

where $\eta' \equiv e^\eta - 1$, provided that $\bar{\rho}_i > \eta'$.

Proof. Since permuting the columns does not change the singular values or left singular vectors, we consider the matrix

$$\hat{B} \equiv \begin{pmatrix} \hat{B}_{11} & \hat{B}_{12} \\ & \hat{B}_{22} \end{pmatrix} \equiv \begin{pmatrix} (\tilde{B}_{11} \quad \tau I) & (\tilde{B}_{12} \quad 0_{k\times(n-k)}) \\ & (\tilde{B}_{22} \quad \tau I) \end{pmatrix}$$

instead of $(\tilde{B} \quad \tau I)$. Since $\hat{B}_{11}$ has full rank, the generalized inverse $\hat{B}_{11}^\dagger$ exists and $\hat{B}_{11}\hat{B}_{11}^\dagger = I$. Letting $X = -\hat{B}_{11}^\dagger\hat{B}_{12}$, we have

$$\|X\| \leq \|\hat{B}_{11}^\dagger\|\,\|\hat{B}_{12}\| = \frac{\|\tilde{B}_{12}\|}{\sigma_{min}[\hat{B}_{11}]} = \frac{\|\tilde{B}_{12}\|}{\sqrt{\sigma_{min}[\tilde{B}_{11}]^2 + \tau^2}} \equiv \eta.$$

Thus both bounds follow from Lemma 5.1 and the fact that B and $\hat{B}$ have the same singular values and left singular vectors. □

Fernando and Parlett [11] prove the slightly weaker result

$$|\sigma_i' - \sigma_i| \leq \sigma_i\,\frac{\|\tilde{B}_{12}\|}{\sigma_{min}[\tilde{B}] + \tau}$$

in the special case where $\tilde{B}$ is bidiagonal.

6. Extensions. In this paper we have demonstrated that the traditional perturbation results and a simple, yet powerful class of perturbations lead to elementary proofs of a wide variety of relative perturbation and deflation results. There are several other applications of this approach, including:

- simple proofs of the relative perturbation results in Demmel and Veselic [7] for the eigenvalues and eigenvectors of a positive definite matrix and the singular values and singular vectors of a general matrix [19].
- bounds for the perturbation in an invariant subspace of dimension greater than one in terms of a relative gap;
- relative deflation results for the eigenproblem.

The latter two applications will be presented in a forthcoming paper.

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