Chapter 1
Relative Perturbation Bounds for Eigenspaces and Singular Vector Subspaces

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Abstract

We present new bounds on the change in the eigenvectors of a real symmetric matrix A due to a perturbation of the form $A + \delta A = D^T A D$, where D is real and nonsingular; and in the singular vectors of a real matrix B due to a perturbation of the form $B + \delta B = D_L B D_R$, where D_L and D_R are real and nonsingular. These classes of perturbations include componentwise relative perturbations of bidiagonal and biacyclic matrices and symmetric tridiagonal matrices with zero diagonal, as well as perturbations that annihilate off-diagonal elements. In particular, we bound the angle between the space spanned by an eigenvector (singular vector) of the perturbed matrix and an eigenspace (singular vector subspace) of the original matrix in terms of the *relative*, rather than absolute, gap in the eigenvalues or singular values. We also present bounds on the angle between subspaces.

1 The Problem

Let A be an $n \times n$ real symmetric matrix with eigendecomposition $A = X \Lambda X^T$, where X is orthogonal and Λ is diagonal. Assume that the eigenvalues of A are grouped so that

$$\Lambda = \begin{array}{c c} & \begin{array}{cc} k & n-k \end{array} \\ \begin{array}{c} k \\ n-k \end{array} & \begin{pmatrix} \Lambda_1 & \\ & \Lambda_2 \end{pmatrix} \end{array} \qquad \text{and} \qquad X = \begin{array}{c} \begin{array}{cc} k & n-k \end{array} \\ \begin{pmatrix} X_1 & X_2 \end{pmatrix} \end{array}.$$

For example, Λ_1 might correspond to a cluster of nearly coincident eigenvalues that is well-separated from the other eigenvalues (those corresponding to Λ_2).

Now let $A + \delta A$ be a real symmetric perturbation of A with eigenvalue λ' and corresponding unit eigenvector x'; i.e.,

$$(A + \delta A)x' = \lambda' x', \qquad \text{with} \quad \|x'\| = 1,$$

where $\|\cdot\|$ denotes the two-norm. The traditional *absolute* error bound for the angle $0 \leq \theta' \leq \pi/2$ between the spaces spanned by x' and by the columns of X_1 is given in the following lemma, which is a special case of Davis and Kahan's second generalized $\sin\theta$ theorem [1, Theorem 6.2].

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LEMMA 1.1.
$$\sin\theta' \le \frac{\|\delta A\|}{\min_j |(\Lambda_2)_{jj} - \lambda'|}.$$

Proof. Write
$$x' = X_1 c + X_2 s, \qquad \text{where} \quad c = X_1^T x' \quad \text{and} \quad s = X_2^T x'.$$
Then $\|c\| = \cos\theta'$, so that $\|s\| = \sin\theta'$ (see [3, Section 12.4.3]). Substituting the expression for x' into $(A + \delta A)x' = \lambda' x'$ gives
$$\delta A \cdot x' = (\lambda' I - A)(X_1 c + X_2 s) = X_1(\lambda' I - \Lambda_1)c + X_2(\lambda' I - \Lambda_2)s.$$
Thus
$$\begin{aligned} \|\delta A \cdot x'\| &= \|X^T \cdot \delta A \cdot x'\| = \left\| \begin{pmatrix} (\lambda' I - \Lambda_1)c \\ (\lambda' I - \Lambda_2)s \end{pmatrix} \right\| \\ &\ge \|(\lambda' I - \Lambda_2)s\| \ge \min_j |(\Lambda_2)_{jj} - \lambda'| \, \|s\|. \end{aligned}$$
The result then follows immediately. ☐

Note that the bound depends upon the *absolute* gap between λ' and the eigenvalues corresponding to Λ_2.

In this paper we consider perturbations of the form $A + \delta A = D^T A D$, where D is real and nonsingular. This class includes componentwise relative perturbations of a tridiagonal matrix with zero diagonal and perturbations that annihilate off-diagonal elements. We give a new bound for $\sin\theta'$ that depends upon the *relative gap*[1]
$$\rho' \equiv \begin{cases} \min_j \dfrac{|(\Lambda_2)_{jj} - \lambda'|}{|\lambda'|} & \text{if } \lambda' \ne 0 \\ \infty & \text{otherwise} \end{cases}$$
between λ' and the eigenvalues corresponding to Λ_2. In §2 we prove this basic result, and in §3 we extend it to the angle between a k-dimensional invariant subspace of the perturbed matrix and the space spanned by the columns of X_1.

In §4 we state the corresponding results for the singular vectors of a matrix B for perturbations of the form $B + \delta B = D_L B D_R$, where D_L and D_R are real and nonsingular. This class of perturbations includes componentwise relative perturbations of bidiagonal and biacyclic matrices, as well as perturbations that annihilate off-diagonal elements.

These results extend to subspaces our previous work [2] on relative perturbation techniques for the eigenvalue and singular value problems.

2 The Angle Between an Eigenvector and an Eigenspace

Let $A + \delta A = D^T A D$, where D is real and nonsingular. Let λ' be an eigenvalue of $A + \delta A$ with corresponding unit eigenvector x'. In this section we bound the angle $0 \le \theta' \le \pi/2$ between the spaces spanned by x' and by the columns of X_1 in terms of the relative gap ρ'.

THEOREM 2.1. *Let*
$$\alpha \equiv \|D^{-T} D^{-1} - I\| \qquad \textit{and} \qquad \beta \equiv \|D - I\|.$$

[1] This definition is slightly different from that in [2] and leads to a stronger result when $k = 1$ (see Remark 2.1).

Then

$$\sin\theta' \le \frac{\alpha}{\rho'} + \beta.$$

Proof. Let

$$M = A - \lambda' I, \qquad \delta M = \lambda'(I - D^{-T}D^{-1}), \qquad \text{and} \qquad \bar{x} = Dx'/\|Dx'\|,$$

so that

$$MX_1 = X_1(\Lambda_1 - \lambda' I) \qquad \text{and} \qquad (M + \delta M)\bar{x} = 0\bar{x}.$$

Let $0 \le \bar{\theta} \le \pi/2$ be the angle between the spaces spanned by $\bar{x}$ and by the columns of X_1. By Lemma 1.1,

$$\sin\bar{\theta} \le \frac{\|\delta M\|}{\min\limits_j |(\Lambda_2)_{jj} - \lambda'|} \le \frac{|\lambda'|\alpha}{\min\limits_j |(\Lambda_2)_{jj} - \lambda'|} \le \frac{\alpha}{\rho'}.$$

If $\|Dx'\| \le 1$, then

$$\begin{aligned}\sin\theta' &= \|X_2^T x'\| \le \|X_2^T D x'\| + \|X_2^T(Dx' - x')\| \\ &\le \|Dx'\|\,\|X_2^T \bar{x}\| + \|(D - I)x'\| \le \|X_2^T\bar{x}\| + \|D - I\| = \sin\bar{\theta} + \beta;\end{aligned}$$

otherwise

$$\begin{aligned}\sin\theta' &= \|X_2^T x'\| \le \|X_2^T\bar{x}\| + \|X_2^T(\bar{x} - x')\| \le \sin\bar{\theta} + \|\bar{x} - x'\| \\ &\le \sin\bar{\theta} + \|Dx' - x'\| \le \sin\bar{\theta} + \|D - I\| \le \sin\bar{\theta} + \beta.\end{aligned}$$

The result then follows immediately. □

Now assume that λ' is an approximation to some $(\Lambda_1)_{ii}$ in the sense that there exists an l such that λ' is the l^{th} largest eigenvalue of $A + \delta A$ and $(\Lambda_1)_{ii}$ is the l^{th} largest eigenvalue of A. In this case we can bound $\sin\theta'$ in terms of the relative gap

$$\rho_i \equiv \begin{cases} \min\limits_j \dfrac{|(\Lambda_2)_{jj} - (\Lambda_1)_{ii}|}{|(\Lambda_1)_{ii}|} & \text{if } (\Lambda_1)_{ii} \ne 0 \\ \infty & \text{otherwise}\end{cases}$$

between $(\Lambda_1)_{ii}$ and the eigenvalues corresponding to Λ_2. The following result generalizes Theorem 2.2 in [2].

COROLLARY 2.1. *Let*

$$\beta \equiv \|D - I\|, \qquad \gamma \equiv \|D^T D - I\|, \qquad \textit{and} \qquad \delta \equiv \|D^T D\|\,\|D^{-T}D^{-1} - I\|.$$

Then

$$\sin\theta' \le \frac{\delta}{\rho_i - \gamma} + \beta,$$

provided that $\rho_i > \gamma$.

Proof. By [2, Theorem 2.1],

$$|\lambda'| \le |(\Lambda_1)_{ii}|\,\|D^T D\| \qquad \text{and} \qquad |\lambda' - (\Lambda_1)_{ii}| \le |(\Lambda_1)_{ii}|\,\|D^T D - I\|.$$

But

$$\begin{aligned}\min_j |(\Lambda_2)_{jj} - \lambda'| &\ge \min_j |(\Lambda_2)_{jj} - (\Lambda_1)_{ii}| - |\lambda' - (\Lambda_1)_{ii}| \\ &\ge \min_j |(\Lambda_2)_{jj} - (\Lambda_1)_{ii}| - |(\Lambda_1)_{ii}|\,\gamma,\end{aligned}$$

provided that $\rho_i > \gamma$. Thus, $\rho_i < \rho'$, and the result follows from Theorem 2.1. □

REMARK 2.1. *In the special case $k = 1$, Theorem 2.1 is stronger than Theorem 2.2 in [2] or Corollary 2.1, because it does not assume that λ' is an approximation to some $(\Lambda_1)_{ii}$. However, this stronger assumption should not pose any problems in practice. Unless Λ_1 represents a poorly separated cluster, the separation between Λ_1 and Λ_2 will be larger than the size of the perturbation δA, so that the eigenvalues will not get jumbled up in the transition from A to $A + \delta A$.*

3 Angles Between Eigenspaces of the Same Dimension

Let $A + \delta A = D^T A D$, where D is real and nonsingular. Let Λ_1' be a $k \times k$ diagonal matrix whose diagonal entries are eigenvalues of $A + \delta A$, and let X_1' be the $n \times k$ matrix of corresponding eigenvectors; i.e.,

$$(A + \delta A) X_1' = X_1' \Lambda_1', \qquad \text{where } X_1' \text{ has orthonormal columns.}$$

Let $0 \leq \theta \leq \pi/2$ be the angle between the spaces spanned by the columns of X_1' and by the columns of X_1, so that

$$\sin\theta = \|X_2^T X_1'\|$$

(see [3, Section 12.4.3]). In this section we bound $\sin\theta$ in terms of the relative gap

$$\rho \equiv \begin{cases} \min\limits_{i,j} \dfrac{|(\Lambda_1)_{ii} - (\Lambda_2)_{jj}|}{|(\Lambda_1)_{ii}|} & \text{if } (\Lambda_1)_{ii} \neq 0, \quad 1 \leq i \leq k \\ \infty & \text{otherwise} \end{cases},$$

which represents the relative separation between the two parts of the spectrum of A.

THEOREM 3.1. *Assume that the eigenvalues in Λ_1 and Λ_1' are ordered according to size, and that for each j there exists an l_j such that $(\Lambda_1')_{jj}$ is the l_j^{th} eigenvalue of $A + \delta A$ and $(\Lambda_1)_{jj}$ is the l_j^{th} eigenvalue of A. Let*

$$\beta \equiv \|D - I\|, \qquad \gamma \equiv \|D^T D - I\|, \qquad \textit{and} \qquad \delta \equiv \|D^T D\|\, \|D^{-T} D^{-1} - I\|.$$

Then

$$\sin\theta \leq \sqrt{k}\left(\frac{\delta}{\rho - \gamma} + \beta\right),$$

provided that $\rho > \gamma$.

Proof. Write $X_1' = X_1 C + X_2 S$, where $C = X_1^T X_1'$ and $S = X_2^T X_1'$. Then $\sin\theta = \|S\|$. Assume that $\|S e_i\| \equiv \max_{1 \leq j \leq k} \|S e_j\|$, so that $\|S\| \leq \sqrt{k}\|S e_i\|$. Now apply Corollary 2.1 with $\lambda' = (\Lambda_1')_{ii}$ and $x' = X_1' e_i$ to get

$$\sin\theta \leq \sqrt{k}\|S e_i\| \leq \sqrt{k}\left(\frac{\delta}{\rho_i - \gamma} + \beta\right),$$

provided that $\rho_i > \gamma$. But $\rho \leq \rho_i$, and the result follows. $\square$

4 Results for Singular Vector Subspaces

Let B be an $m \times n$ real matrix with singular value decomposition $B = U \Sigma V^T$, where U and V are orthogonal and Σ is diagonal. Assume that the singular values of B are grouped so that

$$\Sigma = \begin{array}{c} \\ k \\ n-k \end{array} \!\!\begin{array}{c} \begin{array}{cc} k & n-k \end{array} \\ \begin{pmatrix} \Sigma_1 & \\ & \Sigma_2 \end{pmatrix} \end{array}, \qquad U = \begin{array}{c} \begin{array}{cc} k & n-k \end{array} \\ \begin{pmatrix} U_1 & U_2 \end{pmatrix} \end{array}, \qquad \text{and} \qquad V = \begin{array}{c} \begin{array}{cc} k & n-k \end{array} \\ \begin{pmatrix} V_1 & V_2 \end{pmatrix} \end{array}.$$

Now let $B + \delta B$ be a perturbation of B of the form $B + \delta B = D_L B D_R$, where D_L and D_R are real and nonsingular. Let σ' be a singular value of $B + \delta B$, and let u' and v' be the corresponding unit left and right singular vectors; i.e.,

$$(B+\delta B)v' = \sigma' u' \qquad \text{and} \qquad (B+\delta B)^T u' = \sigma' v', \qquad \text{with} \quad \|u'\| = \|v'\| = 1.$$

We begin by bounding the angle $0 \le \theta^{u'} \le \pi/2$ between the spaces spanned by u' and by the columns of U_1, and the angle $0 \le \theta^{v'} \le \pi/2$ between between the spaces spanned by v' and by the columns of V_1 in terms of the relative gap

$$\bar{\rho}' \equiv \begin{cases} \min\left\{2,\ \min_j \dfrac{|(\Sigma_2)_{jj} - \sigma'|}{|\sigma'|}\right\} & \text{if } \sigma' \neq 0 \\ \infty & \text{otherwise} \end{cases}.$$

THEOREM 4.1. *Let*

$$\bar{\alpha} \equiv \max\{\|D_L^{-1} D_L^{-T} - I\|, \|D_R^{-T} D_R^{-1} - I\|\} \quad \textit{and} \quad \bar{\beta} \equiv \max\{\|D_L^T - I\|, \|D_R - I\|\}.$$

Then

$$\max\{\sin\theta^{u'}, \sin\theta^{v'}\} \le \sqrt{2}\left(\frac{\bar{\alpha}}{\bar{\rho}'} + \bar{\beta}\right).$$

Now assume that σ' is an approximation to some $(\Sigma_1)_{ii}$ in the sense that there exists an l such that σ' is l^{th} largest singular value of $B + \delta B$ and $(\Sigma_1)_{ii}$ is the l^{th} largest singular value of B. In this case we can bound $\sin\theta^{u'}$ and $\sin\theta^{v'}$ in terms of the relative gap

$$\bar{\rho}_i \equiv \begin{cases} \min\left\{2,\ \min_j \dfrac{|(\Sigma_2)_{jj} - (\Sigma_1)_{ii}|}{|(\Sigma_1)_{ii}|}\right\} & \text{if } (\Sigma_1)_{ii} \neq 0 \\ \infty & \text{otherwise} \end{cases}$$

between $(\Sigma_1)_{ii}$ and the singular values corresponding to Σ_2. The following result generalizes Theorem 3.3 in [2].

COROLLARY 4.1. *Let*

$$\bar{\beta} \equiv \max\{\|D_L^T - I\|, \|D_R - I\|\}, \qquad \bar{\gamma} \equiv \max\{\|D_L D_L^T - I\|, \|D_R^T D_R - I\|\},$$

and

$$\bar{\delta} \equiv \max\{\|D_L D_L^T\|, \|D_R^T D_R\|\} \cdot \max\{\|D_L^{-1} D_L^{-T} - I\|, \|D_R^{-T} D_R^{-1} - I\|\}.$$

Then

$$\max\{\sin\theta^{u'}, \sin\theta^{v'}\} \le \sqrt{2}\left(\frac{\bar{\delta}}{\bar{\rho}_i - \bar{\gamma}} + \bar{\beta}\right).$$

The analog of Theorem 3.1 is similar.

References

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