THE LACK OF INFLUENCE OF THE RIGHT-HAND SIDE ON THE ACCURACY OF LINEAR SYSTEM SOLUTION

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Abstract. It is commonly believed that a fortunate right-hand side b can significantly reduce the sensitivity of a system of linear equations $Ax = b$. We show, both theoretically and experimentally, that this is not true when the system is solved (in floating point arithmetic) with Gaussian elimination or the QR factorization: The error bounds essentially do not depend on b; and the error itself seems to depend only weakly on b. Our error bounds are exact (rather than first-order); they are tight, and they are stronger than the bound of Chan and Foulser.

We also present computable lower and upper bounds for the relative error. The lower bound gives rise to a stopping criterion for iterative methods that is better than the relative residual. This is because the relative residual can be much larger, and it may be impossible to reduce it to a desired tolerance.

Key words. linear system, right-hand side, condition number, backward error, stopping criterion

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1. Introduction.

When a system of linear equations $Ax = b$ is solved numerically, the accuracy of the computed solution generally depends on the sensitivity of the linear system to perturbations. In this paper we examine how the right-hand side b affects the sensitivity of the linear system and the error estimates.

Suppose the matrix A is non-singular and $b \neq 0$, so $x \neq 0$. Then the accuracy of a computed solution $\hat{x}$ can be determined from the norm-wise relative error $\|x-\hat{x}\|/\|x\|$. This error is often estimated from an upper bound of the form

$$\|x - \hat{x}\|/\|x\| \leq \text{condition number} * \text{backward error}.$$

The 'backward error', very informally, reflects the accuracy of the input data A and b, and how well the computed solution $\hat{x}$ solves the linear system $Ax = b$. (This is in contrast to the 'forward error' $\|x - \hat{x}\|/\|x\|$, which reflects the accuracy of the output.) The condition number is interpreted as a measure for the sensitivity of the linear system because it amplifies the inaccuracy of the input. The condition number in most error bounds depends only on A but not b.

A stable, accurate linear system solver, such as Gaussian elimination with partial pivoting or the QR factorization, usually produces a backward error that is proportional to, among other factors, the product of the machine precision ϵ_{mach} and a slowly growing function of the matrix size n. For instance, in IEEE single precision where $\epsilon_{mach} \approx 10^{-7}$, the backward error cannot be smaller than 10^{-7} because the

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data A and b cannot be represented more accurately. For a linear system with condition number of about 10^7, the above error bound is on the order of one. In this case we should be prepared to expect a complete loss of accuracy in at least one component of $\hat{x}$.

In this paper we assume that $Ax = b$ is solved by a general-purpose linear system solver: Gaussian elimination with partial pivoting or the QR factorization. Excluding from consideration special-purpose linear system solvers designed to exploit structure in the matrix, such as the fast sine solvers considered in [5]; or Toeplitz, Cauchy or Vandermonde solvers [4] relieves us from assuming additional properties of the backward error.

The question we are trying to answer is whether the error bound depends on properties of the right-hand side b. Why is this important? Of course, the influence of b is important when the linear system is ill-conditioned (i.e. when the condition number is on the order of $1/\epsilon_{mach}$). If a fortunate right-hand side could decrease the condition number, then the error may decrease. This means the computed solution associated with a fortunate right-hand side is more accurate than one associated with an unfortunate right-hand side.

But the influence of b is also important for general linear systems as they become large, because condition numbers usually grow with n. Although a large linear system may look well-conditioned because the condition number is merely a small multiple of n, it may be ill-conditioned on our machine because the condition number is on the order of $1/\epsilon_{mach}$. According to the above error bound, the matrix size n must be significantly smaller than $1/\epsilon_{mach}$ if $\hat{x}$ is to have any accuracy at all. But if a fortunate right-hand side could make the condition number very small, then this soothing effect would become more pronounced as n increases. This implies that we could solve linear systems with fortunate right-hand sides that are much larger than systems with unfortunate right-hand sides.

The paper is organised as follows. Section 2 starts with exact (rather than first-order) residual bounds on the relative error. The condition number in the upper bound is much smaller for some right-hand sides than for others. However, the backward error depends on the condition number. Therefore it is difficult to say anything about the product of condition number and backward error. Section 3 shows that the error bound as a whole does not depend on b. We express the bound in terms of an alternative condition number and backward error that are also independent of b. Section 4 presents a computable, a posteriori version of the error bound; and §5 uses this bound to evaluate stopping criteria for iterative methods. Section 6 expresses the error bound in terms of a third backward error, because this error is the basis for another popular stopping criterion. Section 7 presents Chan and Foulser's 'effective condition number', and shows that it is weaker than our condition number from §2. Section 8 shows that the relative error does not behave like the error bound, because it appears to be weakly dependent on b. After the conclusion in §9, Appendix A briefly discusses how the numerical experiments were carried out.

2. Dependence on the Right-Hand Side. We present a residual bound for the relative error that contains a condition number dependent on the right-hand side. This condition number can be significantly smaller than the traditional matrix condition number.

Let A be a $n \times n$ non-singular, complex matrix and $b \neq 0$ be a $n \times 1$ complex vector. Then the system of linear equations $Ax = b$ has the exact solution $x \neq 0$. We measure the accuracy of a computed solution $\hat{x}$ by means of the norm-wise relative error $\|x-\hat{x}\|/\|x\|$, where $\|\cdot\|$ is a p-norm. The relative error can be bounded in terms of the residual

$$r \equiv b - A\hat{x}$$

as follows [11, Theorem 7.2]:

$$\frac{1}{\|A\|}\,\frac{\|r\|}{\|x\|} \leq \frac{\|x-\hat{x}\|}{\|x\|} \leq \|A^{-1}\|\,\frac{\|r\|}{\|x\|}. \tag{2.1}$$

These are exact, as opposed to first-order, bounds. Although the bounds to follow in §2 and §3 represent different interpretations, they are all identical to (2.1).

2.1. Interpretation of the Bounds. Writing

$$A\hat{x} = b - r$$

shows that $\hat{x}$ is the solution to a linear system with perturbed right-hand side. Thus, we compensate for the error in $\hat{x}$ by changing the right-hand side. Expressing (2.1) in terms of the corresponding backward error $\|r\|/\|b\|$ gives [16, Theorem III.2.13]:

$$\frac{\kappa(A,b)}{\kappa(A)}\,\frac{\|r\|}{\|b\|} \leq \frac{\|x-\hat{x}\|}{\|x\|} \leq \kappa(A,b)\,\frac{\|r\|}{\|b\|}, \tag{2.2}$$

where

$$\kappa(A) \equiv \|A\|\,\|A^{-1}\|$$

is the traditional matrix condition number; and

$$\kappa(A,b) \equiv \frac{\|A^{-1}\|\,\|b\|}{\|x\|} = \frac{\|A^{-1}\|\,\|b\|}{\|A^{-1}b\|}$$

is a condition number that depends on the right-hand side. In particular, $\kappa(A,b)$ is invariant under scalar multiplication of b, but it does depend on the direction of b:

$$1 \leq \kappa(A,b) \leq \kappa(A). \tag{2.3}$$

For instance, consider the case when b is a multiple of a left singular vector u_k of A. If σ_k is a corresponding singular value and v_k a corresponding right singular vector (see §7), i.e.

$$Av_k = \sigma_k u_k, \qquad \|u_k\|_2 = \|v_k\|_2 = 1,$$

then the two-norm version of $\kappa(A,b)$ equals

$$\kappa_2(A,b) = \frac{\sigma_k}{\min_i \sigma_i}.$$

Thus, a right-hand side b lying along a singular vector with a small σ_k has a smaller condition number than a b lying along a singular vector with a large σ_k. In general, when $\kappa(A) \gg 1$ then there are right-hand sides b for which $\kappa(A,b)$ is significantly smaller than $\kappa(A)$.

The bounds on $\kappa(A,b)$ imply the traditional residual bounds for the relative error [15, Theorem 4.3]:

$$\frac{1}{\kappa(A)}\,\frac{\|r\|}{\|b\|} \leq \frac{\|x-\hat{x}\|}{\|x\|} \leq \kappa(A)\,\frac{\|r\|}{\|b\|}. \tag{2.4}$$

If the error in $\hat{x}$ can be attributed solely to input perturbations of the right-hand side then $\|r\|/\|b\|$ reflects the accuracy of the input data and it is an appropriate measure for the size of these perturbations. Hence there are right-hand sides for which the bounds (2.2) are tighter than the traditional bounds (2.4). Therefore the error bounds depend on b; and a fortunate right-hand side can reduce the sensitivity of the linear system and increase the accuracy.

2.2. Related Work. The potentially soothing effect of the right-hand side has been known for some time.

In the context of special-purpose linear system solvers designed to exploit structure in a matrix, a fortunate right-hand side can significantly reduce the sensitivity of the linear system. This is the case, for instance, when A is a triangular M-matrix, all components of b are non-negative and $Ax = b$ is solved by backsubstitution [9, Theorem 3.5]; or when A is a Vandermonde matrix derived from real, non-negative points arranged in increasing order, the elements of b alternate in sign, and $Ax = b$ is solved by the Björck-Pereyra algorithm [8, §3]. A component-wise infinity-norm version[1] of $\kappa(A,b)$,

$$\|\,|A^{-1}|\,|b|\,\|_\infty/\|x\|_\infty,$$

is introduced in [4, §4] to explain the high relative accuracy of certain algorithms for solving linear systems whose matrix is a totally-positive Cauchy or Vandermonde matrix.

In the context of general purpose algorithms, the situation is not as clear due to the lack of hard results. For instance, when $\kappa(A,b) = 1$ and $\|b\| = 1$ then $\|x\| = \|A^{-1}\|$. In this case Stewart says: 'the solution of the system $Ax = b$ reflects the condition of A' [16, p 126]; and 'a problem that reflects the condition of A is insensitive to perturbations in b, even if $\kappa(A)$ is large' [15, p 194].

Chan and Foulser [5] define an 'effective condition number' [5, Theorem 1] that is small when b is related to A in a special way. They conclude that for appropriate right-hand sides 'the sensitivity of x can be substantially smaller than that predicted by $\kappa(A)$ alone' [5, p 963] (However, in §7 we show that the effective condition number is never smaller than $\kappa(A,b)$ and can, in fact, be much larger than $\kappa(A)$.) In the context of linear systems arising from a boundary collocation method for solving Laplace's equation on a two-dimensional domain, Christiansen and Hansen [7] confirm that 'the ordinary condition number is orders of magnitude larger than the effective condition number' [7, §5.1].

Thus there is evidence that a fortunate right-hand side may be able to reduce the sensitivity of a linear system to perturbations in the right-hand side. Can we therefore conclude that $Ax = b$ is well-conditioned whenever $\kappa(A,b)$ is small – even if $\kappa(A)$ is large?

[1] Here $|A|$ denotes the matrix whose elements are the absolute values of the corresponding elements of A.

2.3. Numerical Experiments. To answer this question, we compute $\|r\|_2/\|b\|_2$ in the two-norm.

We chose sixteen matrices from the MATLAB test matrix suite [10]. The matrices are real and have various properties: non-symmetric, symmetric, indefinite, positive definite, triangular or tridiagonal. The triangular matrices R(Compan) and R(Dorr) are upper triangular matrices from QR factorizations of the matrices Compan and Dorr, respectively.

The order n of a matrix A is determined so that its two-norm condition number $\kappa_2(A)$ lies between 10^5 and 10^7. Thus the matrix orders range from 5 to 1000. The purpose is to push the limits of single precision accuracy (about 10^{-7}): With a condition number of 10^7 and a relative residual on the order of single precision, the upper bound on the traditional relative error (2.4) equals one. This means at least one component of the computed solution $\hat{x}$ may have no correct digits. We designed these extreme cases to see clearly whether a fortunate right-hand side is capable of providing relief in the worst case.

We choose the right-hand sides for the linear systems as follows. Each matrix A is paired up in turn with nine different right-hand sides. To obtain a range of $\kappa_2(A,b)$ values we forced the right-hand sides to lie along three different directions: one direction maximises $\kappa_2(A,b)$: $\kappa_2(A,b)=\kappa_2(A)$; another direction minimises $\kappa_2(A,b)$: $\kappa_2(A,b)=1$; and a third direction falls in between: b is a random vector. For each direction, the right-hand sides come in three different lengths, long: $\|b\|_2=\kappa_2(A)$; short $\|b\|_2=1/\kappa_2(A)$, and unit-norm: $\|b\|_2=1$.

Each linear system was solved by two different direct methods: Gaussian elimination with partial pivoting (GE) and the QR factorization (QR). The solutions were computed in single precision, with machine precision on the order of 10^{-7}. More details about the experiments are given in Appendix A.

In all tables to follow, the first column represents the direction of b (in terms of $\kappa_2(A,b)$) while the second column represents the length of b (in terms of $\|b\|_2$). For each of the nine different right-hand sides, we display the results from GE and from QR.

Tables 2.1, 2.2 and 2.3 show the following: When $\kappa_2(A,b)$ is large then $\|r\|_2/\|b\|_2$ is on the order of machine precision (except for the Chow, Fiedler and Minij matrices, where QR produces $\|r\|_2/\|b\|_2$ as large as 10^{-5}). However when $\kappa_2(A,b)$ is small then $\|r\|_2/\|b\|_2$ is large, usually between 10^{-3} and 10^{-1}.

The numerical experiments suggest that $\|r\|_2/\|b\|_2$ is inversely proportional to $\kappa_2(A,b)$. Since both condition number and backward error depend on b we cannot draw any conclusions about their product, the error bound.

Therefore we forego $\|r\|/\|b\|$ as a backward error and $\kappa(A,b)$ as a condition number, and look for alternatives.

3. Independence From the Right-Hand Side. We show that the lower and upper bounds (2.1) essentially do not depend on the direction of b when $\hat{x}$ is computed by Gaussian elimination or QR factorization. We rewrite the bounds in terms of a condition number and a backward error that are independent of b. We also explain why $\|r\|/\|b\|$ varies with $\kappa(A,b)$.

3.1. Another Interpretation of the Bounds. We ended up with $\|r\|/\|b\|$ as a backward error because we multiplied and divided the bounds in (2.1) by $\|b\|$. The result (2.2) is a somewhat arbitrary separation of (2.1) into backward error and condition number. If we focus instead on the bounds (2.1) as a whole then an obvious choice for backward error is the lower bound

$$\eta \equiv \frac{\|r\|}{\|A\| \, \|x\|}.$$

This makes sense because unless η is small, the relative error isn't going to be small. Expressing (2.1) in terms of η allows us to bracket the relative error in terms of η and a condition number independent of b,

$$\eta \leq \frac{\|x - \hat{x}\|}{\|x\|} \leq \kappa(A)\, \eta. \tag{3.1}$$

The numerical experiments below suggest that η is essentially independent of b.

3.2. Numerical Experiments. We compute $\eta_2 \equiv \|r\|_2/(\|A\|_2 \, \|x\|_2)$, the two-norm version of η.

Tables 3.1, 3.2 and 3.3 show the following: Regardless of $\kappa_2(A, b)$, η_2 tends to be on the order of machine precision (except for the Chow, Fiedler and Minij matrices where QR produces values for η_2 as large as 10^{-5}). Thus, Gaussian elimination and QR factorization produce solutions whose backward error η_2 is usually on the order of machine precision.

We conclude that in the case of Gaussian elimination and QR factorization the bounds (3.1) are essentially independent of b. The independence of general-purpose algorithms from the right-hand side is also confirmed in [4, §5].

3.3. Relation Between Backward Errors. To reconcile the two different interpretations, (2.2) and (3.1), of the bounds (2.1) we relate the backward errors $\|r\|/\|b\|$ and η. The relation was already derived in [6, p 99] and is alluded to in [11, p 342]:

$$\frac{\|r\|}{\|b\|} = \frac{\kappa(A)}{\kappa(A, b)}\, \eta. \tag{3.2}$$

This confirms the observation in §2.3 that $\|r\|/\|b\|$ is inversely proportional to $\kappa(A, b)$, provided η does not depend on b. Relation (3.2) implies together with (2.3):

$$\eta \leq \frac{\|r\|}{\|b\|} \leq \kappa(A)\, \eta. \tag{3.3}$$

That is, when $\kappa(A, b)$ is maximal, $\|r\|/\|b\|$ can be as small as machine precision. But when $\kappa(A, b)$ is minimal then $\|r\|/\|b\|$ can be large because it hides the condition number inside.

Therefore, η appears to be preferable as a backward error over $\|r\|/\|b\|$.

4. Computable Error Bounds. We present computable, a posteriori error bounds that do not depend on the direction of b, when $Ax = b$ is solved by Gaussian elimination or QR factorization. The computable version of η is optimal in a well-defined sense.

To obtain bounds that are computable, we measure the relative error instead with regard to the computed solution:

$$\frac{1}{\|A\|} \frac{\|r\|}{\|\hat{x}\|} \leq \frac{\|x - \hat{x}\|}{\|\hat{x}\|} \leq \|A^{-1}\| \, \frac{\|r\|}{\|\hat{x}\|}. \tag{4.1}$$

Expressing (4.1) in terms of the computable version of η,

$$\hat{\eta} \equiv \frac{\|r\|}{\|A\|\,\|\hat{x}\|},$$

yields an interval for the relative error [16, Theorem III.2.16]:

$$\hat{\eta} \leq \frac{\|x-\hat{x}\|}{\|\hat{x}\|} \leq \kappa(A)\,\hat{\eta}. \tag{4.2}$$

The numerical experiments below confirm that $\hat{\eta}$ is as good a measure of accuracy as η.

4.1. Numerical Experiments. We compute $\hat{\eta}_2$, the two-norm version of $\hat{\eta}$.

Tables 4.1, 4.2 and 4.3 show the following: Regardless of $\kappa_2(A,b)$, $\hat{\eta}_2$ is on the order of machine precision (except for the Chow and Fiedler matrices where QR produces values for $\hat{\eta}_2$ as large as 10^{-6}). In case of the Minij matrix, $\hat{\eta}_2$ is on the order machine precision while η_2 can be as large as 10^{-5}. Thus, Gaussian elimination and the QR factorization tend to produce a computed solution whose backward error $\hat{\eta}_2$ is on the order of machine precision.

We conclude that in case of Gaussian elimination and QR factorization the bounds (4.2) are essentially independent of b.

4.2. Minimal Matrix Backward Error. Another justification for the bounds (4.1) is the optimality of $\hat{\eta}$ in the following sense: $\hat{\eta}$ represents the best possible (norm-wise) backward error when perturbations are confined to the matrix.

Whenever $\hat{x} \neq 0$ we can write $(A+E_0)\hat{x} = b$ where [16, Theorem III.2.16]

$$\|E_0\| = \|r\|/\|\hat{x}\|. \tag{4.3}$$

Thus $\hat{x}$ is the solution to a linear system with a perturbed matrix. Here we compensate for the error in $\hat{x}$ by changing the matrix. Moreover, among all E satisfying $(A+E)\hat{x} = b$ there is a matrix E_0 in (4.3) with minimal norm, i.e. $\|E_0\| \leq \|E\|$. In case of the two-norm, for instance, one can choose $E_0 = r\hat{x}^*/\|\hat{x}\|_2^2$. Therefore,

$$\hat{\eta} = \|E_0\|/\|A\|$$

is the smallest norm-wise matrix backward error.

Moreover, $\hat{\eta}$ has a similar relation to $\|r\|/\|b\|$ as η:

$$\frac{\|r\|}{\|b\|} = \frac{\|A\|\,\|\hat{x}\|}{\|b\|}\,\hat{\eta}. \tag{4.4}$$

Consequently, $\|r\|/\|b\| \gg \hat{\eta}$ whenever $\|b\| \ll \|A\|\,\|\hat{x}\|$; which means $\|r\|/\|b\|$ is going to be large whenever $\hat{x}$ reflects the condition of A. Arioli, Duff and Ruiz confirm this by observing that 'even an $\hat{x}$ that is a good approximation to x can have a large residual' [2, p 139].

4.3. Bounds on $\hat{\eta}$. The numerical experiments in §4.1 provide a heuristic justification why $\hat{\eta}$ is small. A theoretical justification comes from the following round-off error bounds.

Gaussian elimination with partial pivoting computes a solution $\hat{x}$ for a system $(A+E)\hat{x} = b$ whose matrix perturbation in the infinity-norm is bounded by [11, Theorem 9.5]

$$\frac{\|E\|_\infty}{\|A\|_\infty} \leq \frac{2n^3\rho}{1-n\epsilon}\,\epsilon,$$

where ρ is the growth factor in Gaussian elimination, and ϵ is the machine precision. Unless ρ is large, $\hat{\eta}$ is small in the infinity-norm:

$$\hat{\eta}_\infty \equiv \frac{\|E_0\|_\infty}{\|A\|_\infty} \leq \frac{\|E\|_\infty}{\|A\|_\infty} \leq \frac{2n^3\rho}{1-n\epsilon}\,\epsilon.$$

The QR factorization computes a solution $\hat{x}$ for a system $(A+E)\hat{x}=b$ whose matrix perturbation is bounded by [11, inequalities (18.7)]

$$\frac{\|E\|_2}{\|A\|_2} \leq \frac{cn^3}{1-cn\epsilon}\,\epsilon,$$

where c is a small positive integer. Again, $\hat{\eta}$ is small in the two-norm:

$$\hat{\eta}_2 \equiv \frac{\|E_0\|_2}{\|A\|_2} \leq \frac{cn^3}{1-cn\epsilon}\,\epsilon.$$

The fact that $\hat{\eta}$ is usually small is applied in the following section to determine a realistic stopping criterion for iterative methods.

5. Stopping Criteria for Iterative Methods. An iterative method solves a linear system by computing a succession of iterates. The method terminates once an iterate satisfies a stopping criterion. Popular stopping criteria require the residual to be sufficiently small. For instance two such stopping criteria are [11, §16.5],

$$\|r\| \leq tol\,\|b\|, \qquad \|r\| \leq tol\,\|A\|\,\|\hat{x}\|,$$

where $\hat{x}$ is the current iterate and tol is a user-supplied tolerance. The first criterion requires that $\|r\|/\|b\|$ should not exceed tol, while the second requires that $\hat{\eta}$ should not exceed tol.

The first criterion can be harder to satisfy than the second [11, §16.5]. To see this, suppose an iterate $\hat{x}$ satisfies $\|r\| \leq tol\,\|A\|\,\|\hat{x}\|$. This implies

$$\frac{\|r\|}{\|b\|} \leq \frac{\|A\|\,\|\hat{x}\|}{\|b\|}\,tol.$$

Hence $\hat{x}$ can be very far from satisfying $\|r\| \leq tol\,\|b\|$ when $\|b\| \ll \|A\|\,\|\hat{x}\|$. This confirms the observation in [2, p 139] that $\|r\|/\|b\|$ 'can be misleading in the case when' $\|b\| \ll \|A\|\,\|x\|$.

Therefore, if at all feasible, stopping criteria in iterative methods should be based on $\hat{\eta}$ rather than on $\|r\|/\|b\|$. Preliminary experiments with the matrices from §§2.3, 3.2 and 4.1 indicate that solutions computed by GMRES [14] do satisfy the criterion based on $\hat{\eta}$.

Issues regarding the appropriate choice of stopping criteria have also been discussed in the context of linear systems arising from discretizations of partial differential equations [3, 13].

6. A Third Interpretation of the Error Bound. We present a third interpretation of the error bounds (2.1) based on a backward error ω that is a mixture of the previous two backward errors. The computable version of ω is optimal in a well-defined sense, and represents the basis for another stopping criterion for iterative methods.

Expressing (2.1) in terms of

$$\omega \equiv \frac{\|r\|}{\|A\|\,\|x\| + \|b\|}$$

gives

$$\left(1 + \frac{\kappa(A,b)}{\kappa(A)}\right)\,\omega \le \frac{\|x - \hat{x}\|}{\|x\|} \le \kappa(A)\,\left(1 + \frac{\kappa(A,b)}{\kappa(A)}\right)\,\omega. \tag{6.1}$$

The definitions of η and ω imply the relation

$$\eta = \left(1 + \frac{\kappa(A,b)}{\kappa(A)}\right)\,\omega, \tag{6.2}$$

and η differs from ω by a factor of at most two:

$$\omega \le \eta \le 2\omega. \tag{6.3}$$

The computable version of ω is

$$\hat{\omega} \equiv \frac{\|r\|}{\|A\|\,\|\hat{x}\| + \|b\|}.$$

It represents the smallest (norm-wise) backward error [11, Theorem 7.1]:

$$\hat{\omega} = \min\{\epsilon : (A+E)\hat{x} = b + f, \quad \|E\| \le \epsilon\|A\|,\ \|f\| \le \epsilon\|b\|\}.$$

Moreover, $\hat{\omega}$ resembles the backward error from [2],

$$\omega_2 \equiv \frac{\|r\|_\infty}{\|A\|_\infty\|\hat{x}\|_1 + \|b\|_\infty}.$$

The experiments in [2, §3] suggest that ω_2 behaves much like $\hat{\eta}$.

The two computable backward errors $\hat{\eta}$ and $\hat{\omega}$ are related by

$$\hat{\eta} = \left(1 + \frac{\|b\|}{\|A\|\,\|\hat{x}\|}\right)\hat{\omega}.$$

Hence

$$\hat{\omega} \le \hat{\eta},$$

and $\hat{\omega}$ is small whenever $\hat{\eta}$ is small. In particular, if $\|b\| \le \|A\|\,\|\hat{x}\|$ then

$$\frac{1}{2}\hat{\eta} \le \hat{\omega} \le \hat{\eta}.$$

Hence the round-off error bounds from §4.3 are also valid for $\hat{\omega}$.

A stopping criterion based on $\hat{\omega}$ terminates an iterative method once an iterate $\hat{x}$ satisfies

$$\|r\| \le \mathit{tol}\,(\|A\|\,\|\hat{x}\| + \|b\|).$$

This criterion is recommended in [11, §16.5], and a version based on ω_2 is recommended in [2, §5].

7. The Effective Condition Number. We present Chan and Foulser's 'effective condition number' κ_{eff} [5] and show that it is weaker than $\kappa_2(A,b)$. That is, the effective condition number is never smaller than $\kappa_2(A,b)$ but can be much larger than $\kappa_2(A)$.

Let $A = U\Sigma V^*$ be a singular value decomposition, where U and V are unitary matrices, Σ is a diagonal matrix, and $*$ denotes the conjugate transpose. The diagonal elements of Σ,

$$0 < \sigma_n \leq \ldots \leq \sigma_1,$$

are the singular values of A. The columns of U and V,

$$U = \begin{pmatrix} u_1 & \ldots & u_n \end{pmatrix}, \qquad V = \begin{pmatrix} v_1 & \ldots & v_n \end{pmatrix},$$

are the left and right singular vectors, respectively.

Partition the columns of U,

$$U_k \equiv \begin{pmatrix} u_k & \ldots & u_n \end{pmatrix}, \qquad 1 \leq k \leq n,$$

and let $P_k \equiv U_k U_k^*$ be the orthogonal projector onto $range(U_k)$. If $P_k b \neq 0$ then

$$\kappa_{eff}(k) \equiv \frac{\sigma_k}{\sigma_n} \left(\frac{\|P_k b\|_2}{\|b\|_2} \right)^{-1}, \qquad 1 \leq k \leq n,$$

is *the kth effective condition number of the linear system $Ax = b$.* Before stating Chan and Foulser's error bound, we show that $\kappa_{eff}(k)$ can never be smaller than $\kappa(A,b)$. A similar inequality is stated in [4, (5.2)].

THEOREM 7.1. *If $P_k b \neq 0$, then*

$$\kappa_{eff}(k) \geq \kappa_2(A,b), \qquad 1 \leq k \leq n.$$

Proof. Partition V and Σ conformally with U,

$$V_k \equiv \begin{pmatrix} v_k & \ldots & v_n \end{pmatrix}, \qquad \Sigma_k \equiv \begin{pmatrix} \sigma_k & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix}.$$

Then $Ax = b$ implies

$$\Sigma_k V_k^* x = c_k, \qquad \text{where} \quad c_k \equiv U_k^* b, \qquad 1 \leq k \leq n,$$

and

$$\|x\|_2 \geq \|c_k\|_2/\sigma_k = \|P_k b\|_2/\sigma_k.$$

Therefore

$$\kappa_2(A,b) = \kappa_2(A)\, \frac{\|b\|_2}{\|A\|_2\, \|x\|_2} \leq \frac{\sigma_1}{\sigma_n}\, \frac{\|b\|_2}{\sigma_1} \left(\frac{\|P_k b\|_2}{\sigma_k} \right)^{-1} = \kappa_{eff}(k).$$

□

The following bound is a direct consequence of (2.2) and Theorem 7.1, and is therefore weaker than (2.2).

COROLLARY 7.2 (Theorem 1 in [5]).

$$\frac{\|x-\hat{x}\|_2}{\|x\|_2} \leq \kappa_{\mathit{eff}}(k)\,\frac{\|r\|_2}{\|b\|_2}, \qquad 1 \leq k \leq n.$$

Chan and Foulser [5, p 964, §1] seem to imply that $\kappa_{\mathit{eff}}(k)$ can be much smaller than the traditional condition number $\kappa_2(A)$ when b is close to the direction of u_n, i.e. when $\|P_n b\|_2/\|b\|_2 \approx 1$. In other cases, however, $\kappa_{\mathit{eff}}(k)$ can be much larger than $\kappa_2(A)$ as the example below illustrates.

REMARK 1. *$\kappa_{\mathit{eff}}(k)$ can be arbitrarily much larger than $\kappa_2(A)$.*

A simple 2×2 matrix illustrates this.

$$A = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

has a condition number $\kappa_2(A) = 3$. A singular value decomposition is $A = U\Sigma V^*$ with

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Sigma = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \qquad V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence

$$P_2 = (\,0 \quad 1\,)^*\,(\,0 \quad 1\,) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

If $Ax = b$ and $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ then the condition number associated with the smallest singular value is

$$\kappa_{\mathit{eff}}(2) = \frac{\sigma_2}{\sigma_2}\left(\frac{\|P_2 b\|_2}{\|b\|_2}\right)^{-1} = \frac{\sqrt{b_1^2+b_2^2}}{|b_2|} = \sqrt{\left|\frac{b_1}{b_2}\right|^2 + 1}$$

Choosing $|b_1|/|b_2|$ large makes $\kappa_{\mathit{eff}}(2)$ arbitrarily large, while $\kappa_2(A) = 3$ remains fixed.

8. The Relative Error. Sections 3-6 argued that the error bounds (2.1) essentially do not depend on the direction of b. Can we conclude that the same is true for the accuracy of the computed solution? That is, does the relative error also not depend on the direction of b?

Suppose for a moment that the accuracy of the computed solution did indeed depend on b. Then the magnitude of the relative error should change with the direction of b. In particular, consider the two linear systems $Ax = b^{(1)}$ with $\kappa(A, b^{(1)}) = 1$, and $Ax = b^{(2)}$ with $\kappa(A, b^{(2)}) = \kappa(A)$. If we also assume that the error behaves in the same way as the bounds (2.2) then we would expect the relative error for the first system to be $\kappa(A)$ times smaller than the error of the second system. Since our matrices are constructed so that $\kappa(A)$ is close to the inverse of machine precision, the relative error for $Ax = b^{(1)}$ should be close to machine precision.

8.1. Numerical Experiments. We compute the two-norm relative error $\|x - \hat{x}\|_2/\|x\|_2$.

Tables 8.1, 8.2 and 8.3 provide only an inconclusive answer. Unlike its lower and upper bounds, the relative error does seem to depend on the right-hand side. But this dependence appears to be weak. It is stronger for some matrices than for others.

Dependence on the Direction of b. The relative errors tend to be smaller when $\kappa_2(A,b)=1$, and larger when $\kappa_2(A,b)=\kappa_2(A)$. The Dorr matrices are an exception: Both GE and QR produce errors for $\kappa_2(A,b)=1$ that can be a factor of ten larger than the errors for $\kappa_2(A,b)=\kappa_2(A)$. Similarly for the Fiedler matrix: GE produces the smallest relative error for the random right-hand side, although it does not have the smallest $\kappa(A,b)$ value.

In case of the Kahan matrix, for instance, the relative errors vary by a factor as high as 10^4. This variation is not too far away from $\kappa_2(A)\approx 7.65*10^6$. Since the Kahan matrix is triangular, no factorization is performed (that's why GE and QR produce exactly the same errors).

Does this mean that triangular matrices exhibit a stronger dependence on the direction of b, or that a factorization can destroy the relation between A and b? To answer these questions we computed the upper triangular factors R(Compan) and R(Dorr) in the QR factorizations of the Compan and Dorr matrices, respectively. In exact arithmetic, the matrix R(A) has the same singular values as A, hence the same matrix condition number. If a factorization did indeed destroy the relation between A and b, then we would expect R(A) to depend more on the right-hand side than A. However, the variation in errors is about the same for the Dorr and R(Dorr) matrices; and the variation in errors for R(compan) is about a factor of ten higher than for the Compan matrix. Thus there is no definite indication that the error of a triangular system depends more strongly on the right-hand side than the error of a general, square system.

Now consider the size of the relative errors when $\kappa_2(A,b)=1$. The relative error for the Kahan matrix is on the order of machine precision, but the error produced by the QR factorization of the Minij matrix is on the order of 10^{-1}, significantly larger than machine precision.

We conclude that the variation in errors for different values of $\kappa_2(A,b)$ is usually much smaller than $\kappa_2(A)$; and that the error is significantly larger than machine precision when $\kappa_2(A,b)=1$. Therefore, the error appears to depend only weakly on the direction of the right-hand side. The independence of the relative error from the right-hand side in the case of Gaussian elimination is also observed in [4, §5].

Dependence on the Length of b. Although the bounds in §2 and §3 are invariant under the length of b, the computable bounds in §4 do change with $\|b\|$. Sometimes the magnitude of the errors changes with $\|b\|$ and sometimes it does not. Usually the variation in errors is limited to a factor of about ten. For some matrices, such as the Fiedler, Dorr and Minij matrices, the magnitude of the errors does not change with $\|b\|$. But in other cases, such as the Clement matrix, the magnitude of the errors can differ by a factor of 100.

Dependence on Algorithms. The relative error also depends on the algorithms. Gaussian elimination produces a smaller relative error than QR: The difference in errors can be as high as a factor of 10^6, e.g., for the Minij matrix when $\kappa_2(A,b)=1$. This could be due to the higher operation count of QR and the larger amount of fill in the triangular factor.

The error for QR is often of the same magnitude as the upper bound (2.1), e.g., for the Minij and Fiedler matrices. Thus the upper bounds for the error are realistic.

9. Conclusion. We have investigated how the direction of the right-hand side b affects the error bounds for a system of linear equations $Ax=b$.

If the error in $\hat{x}$ is due solely to input perturbations of the right-hand side then

$\|r\|/\|b\|$ reflects the accuracy of the input data. The norm-wise relative error can be estimated from the bounds

$$\frac{\kappa(A,b)}{\kappa(A)}\,\frac{\|r\|}{\|b\|} \leq \frac{\|x-\hat{x}\|}{\|x\|} \leq \kappa(A,b)\,\frac{\|r\|}{\|b\|},$$

where the condition number

$$\kappa(A,b) \equiv \frac{\|A^{-1}\|\,\|b\|}{\|x\|}$$

is interpreted as a measure for the sensitivity to perturbations in the right-hand side. The error bounds depend on the right-hand side because a fortunate choice of b can significantly reduce the condition number $\kappa(A,b)$ and may thus increase the accuracy.

If, however, perturbations are not confined exclusively to the right-hand side then $\|r\|/\|b\|$ can be much larger than the inaccuracy in the data and the backward error from a linear system solver. To account for perturbations in the matrix, the error bounds are expressed in terms of $\eta = \|r\|/(\|A\|\,\|x\|)$, where

$$\eta \leq \frac{\|x-\hat{x}\|}{\|x\|} \leq \kappa(A)\,\eta.$$

According to numerical and theoretical evidence, η tends to be on the order of machine precision when $\hat{x}$ is computed by Gaussian elimination with partial pivoting or by the QR factorization. Hence the lower and upper error bounds are essentially independent of b.

Our numerical experiments suggest that the upper error bound is realistic because it is often achieved by the QR factorization.

In the context of iterative methods we recommend $\|r\| \leq tol\,\|A\|\,\|\hat{x}\|$ as a stopping criterion over $\|r\| \leq tol\,\|b\|$. This is because $\|r\|/\|b\|$ is much larger than $\hat{\eta}$ when $\|b\| \ll \|A\|\,\|\hat{x}\|$. Preliminary experiments indicate that GMRES (without preconditioning) can produce solutions $\hat{x}$ that satisfy $\|r\| \leq tol\,\|A\|\,\|\hat{x}\|$ with tol equal to machine precision. However they can be far from satisfying $\|r\| \leq tol\,\|b\|$. A third stopping criterion, $\|r\| \leq tol\,(\|A\|\,\|\hat{x}\| + \|b\|)$, behaves very much like $\|r\| \leq tol\,\|A\|\,\|\hat{x}\|$. Hence it is preferable to $\|r\|/\|b\|$, as well.

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Appendix A. Implementation of the Numerical Experiments. We chose the two-norm because it is easy to determine right-hand sides with particular $\kappa_2(A,b)$ values: Let σ_{min} be the smallest singular value of A and σ_{max} be the largest; and denote the corresponding left and right singular vectors by u_{min}, u_{max}, and v_{min}, v_{max}, respectively. This implies for the smallest singular value

$$Av_{min} = \sigma_{min}u_{min}, \qquad \sigma_{min} = 1/\|A^{-1}\|_2,$$

and for the largest singular value

$$Av_{max} = \sigma_{max}u_{max}, \qquad \sigma_{max} = \|A\|_2.$$

Thus, $\kappa_2(A,b)$ takes on its extreme values when b is a left singular vector associated with an extreme singular value. Therefore, we enforced $\kappa_2(A,b) = \kappa_2(A)$ by choosing

b to be a non-zero multiple of u_{max}, and $\kappa_2(A,b)=1$ by choosing b to be a non-zero multiple of u_{min}.

We generated the matrices and right-hand sides in double precision in MATLAB (version 4.2c) [12] and then converted them to single precision, so that A and b admit exact representations in single precision. The triangular matrices R(Compan) and R(Dorr) were computed from the MATLAB QR factorizations of the matrices Compan and Dorr, respectively. To ensure that the right-hand sides lie along the desired directions, we computed for the unit-norm right-hand sides

$$\kappa_2(A,u_{max}) \approx \kappa_2(A), \qquad \kappa_2(A,u_{min}) \approx 1, \qquad \kappa_2(A,\text{random}).$$

The remaining calculations were done in HP FORTRAN 77 (version 9.16). A computed solution $\hat{x}$ is represented by the single precision solution of the following LAPACK subroutines [1]: SGETRF and SGETRS for Gaussian elimination with partial pivoting; and SGEQRF, SORMQR and STRTRS for the QR factorization. To get a representation for the exact solution x, we extended the single precision versions of A and b to double precision and solved the resulting system in double precision by Gaussian elimination (subroutines DGETRF and DGETRS). Note: The exact representations of A and b in single precision ensure that both $\hat{x}$ and x, are computed from the same input data.

The data in Tables 2.1–8.3, $\|r\|_2/\|b\|_2$, η_2, $\hat{\eta}_2$ and $\|x-\hat{x}\|_2/\|x\|_2$ were computed in double precision, after conversion of single precision quantities to double precision.

All computations were performed on a HP9000 Model 712/60 workstation running HP-UX operating system version E release A.09.05.

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TABLE 2.1

Backward Error $\|r\|_2/\|b\|_2$.

Hilbert, $n = 5$, $\kappa_2(A) = 4.77 \cdot 10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	3e-08	1e-07
	1	1e-08	2e-07
	$\kappa_2(A)$	3e-08	4e-08
1	$1/\kappa_2(A)$	1e-03	1e-02
	1	4e-03	9e-03
	$\kappa_2(A)$	2e-03	1e-02
11.1	$1/\kappa_2(A)$	2e-04	7e-04
	1	4e-04	2e-03
	$\kappa_2(A)$	3e-04	7e-04

Vandermonde, $n = 8$, $\kappa_2(A) = 2.68 \cdot 10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	5e-08	6e-08
	1	2e-08	4e-08
	$\kappa_2(A)$	6e-08	8e-08
1	$1/\kappa_2(A)$	1e-03	4e-03
	1	2e-03	5e-03
	$\kappa_2(A)$	2e-03	4e-03
4	$1/\kappa_2(A)$	2e-04	1e-03
	1	5e-04	1e-03
	$\kappa_2(A)$	1e-04	1e-03

Clement, $n = 40$, $\kappa_2(A) = 2.11 \cdot 10^6$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	4e-08	3e-07
	1	3e-08	2e-07
	$\kappa_2(A)$	3e-08	3e-07
1	$1/\kappa_2(A)$	2e-02	4e-02
	1	5e-02	3e-02
	$\kappa_2(A)$	5e-02	5e-02
3.88	$1/\kappa_2(A)$	1e-02	1e-02
	1	9e-03	9e-03
	$\kappa_2(A)$	1e-02	1e-02

Dorr ($\theta = 0.01$), $n = 50$, $\kappa_2(A) = 1.33 \cdot 10^6$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	4e-08	3e-07
	1	4e-08	4e-07
	$\kappa_2(A)$	3e-08	3e-07
1	$1/\kappa_2(A)$	3e-02	6e-02
	1	3e-02	6e-02
	$\kappa_2(A)$	2e-02	6e-02
1.96	$1/\kappa_2(A)$	1e-02	3e-02
	1	1e-02	3e-02
	$\kappa_2(A)$	1e-02	3e-02

Table 2.2

Backward Error $\|r\|_2/\|b\|_2$, *Continued.*

Compan, $n = 1000$, $\kappa_2(A) = 3.35 \cdot 10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	2e-07	2e-07
	1	9e-07	1e-07
	$\kappa_2(A)$	2e-07	2e-07
1	$1/\kappa_2(A)$	3e-03	1e-03
	1	8e-03	4e-03
	$\kappa_2(A)$	6e-03	5e-03
1.31	$1/\kappa_2(A)$	4e-03	4e-03
	1	3e-03	2e-03
	$\kappa_2(A)$	6e-03	5e-03

Tridiag(-1,2,-1), $n = 1000$, $\kappa_2(A) = 4.06 \cdot 10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	3e-08	5e-07
	1	2e-08	6e-07
	$\kappa_2(A)$	3e-08	5e-07
1	$1/\kappa_2(A)$	6e-03	1e-02
	1	6e-03	1e-02
	$\kappa_2(A)$	6e-03	1e-02
1.28	$1/\kappa_2(A)$	8e-03	2e-02
	1	8e-03	2e-02
	$\kappa_2(A)$	7e-03	2e-02

Chow, $n = 1000$, $\kappa_2(A) = 5.42 \cdot 10^5$
($\alpha = 1.0$, $\delta = 0.5$)

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	6e-07	3e-05
	1	2e-07	3e-05
	$\kappa_2(A)$	3e-07	3e-05
1	$1/\kappa_2(A)$	1e-04	3e-01
	1	1e-04	3e-01
	$\kappa_2(A)$	1e-04	3e-01
99.4	$1/\kappa_2(A)$	3e-06	3e-03
	1	1e-06	3e-03
	$\kappa_2(A)$	2e-06	3e-03

Dorr, $n = 1000$, $\kappa_2(A) = 7.18 \cdot 10^5$
($\theta = 0.1$)

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	3e-08	4e-07
	1	2e-08	5e-07
	$\kappa_2(A)$	3e-08	4e-07
1	$1/\kappa_2(A)$	1e-02	3e-02
	1	1e-02	3e-02
	$\kappa_2(A)$	2e-02	3e-02
1.34	$1/\kappa_2(A)$	9e-03	2e-02
	1	1e-02	2e-02
	$\kappa_2(A)$	1e-02	2e-02

Fiedler, $n = 1000$, $\kappa_2(A) = 6.95 \cdot 10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	6e-07	4e-06
	1	1e-06	5e-06
	$\kappa_2(A)$	5e-07	4e-06
1	$1/\kappa_2(A)$	8e-02	3e-02
	1	8e-02	3e-02
	$\kappa_2(A)$	8e-02	3e-02
3.19	$1/\kappa_2(A)$	3e-03	1e-02
	1	3e-02	1e-02
	$\kappa_2(A)$	3e-02	1e-02

Minij, $n = 1000$, $\kappa_2(A) = 1.62 \cdot 10^6$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	6e-07	5e-05
	1	3e-07	5e-05
	$\kappa_2(A)$	2e-07	5e-05
1	$1/\kappa_2(A)$	2e-05	4e-01
	1	4e-04	4e-01
	$\kappa_2(A)$	6e-05	4e-01
3.19	$1/\kappa_2(A)$	6e-07	3e-01
	1	2e-05	3e-01
	$\kappa_2(A)$	1e-06	3e-01

TABLE 2.3

Backward Error $\|r\|_2/\|b\|_2$ *for Triangular Matrices.*

Kahan, $n = 40$, $\kappa_2(A) = 7.65 \cdot 10^6$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	8e-08	8e-08
	1	7e-08	7e-08
	$\kappa_2(A)$	9e-08	9e-08
1	$1/\kappa_2(A)$	7e-02	7e-02
	1	7e-02	7e-02
	$\kappa_2(A)$	4e-02	4e-02
2.68	$1/\kappa_2(A)$	2e-02	2e-02
	1	3e-02	3e-02
	$\kappa_2(A)$	2e-02	2e-02

R(Compan), $n = 1000$, $\kappa_2(A) = 3.35 \cdot 10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	1e-07	1e-07
	1	3e-07	3e-07
	$\kappa_2(A)$	7e-08	7e-08
1	$1/\kappa_2(A)$	2e-03	2e-03
	1	3e-03	3e-03
	$\kappa_2(A)$	4e-03	4e-03
16.7	$1/\kappa_2(A)$	7e-04	7e-04
	1	1e-03	1e-03
	$\kappa_2(A)$	3e-03	3e-03

R(Dorr) ($\theta = .01$), $n = 1000$, $\kappa_2(A) = 7.18 \cdot 10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	4e-08	4e-08
	1	3e-08	3e-08
	$\kappa_2(A)$	4e-08	4e-08
1	$1/\kappa_2(A)$	1e-02	1e-02
	1	1e-02	1e-02
	$\kappa_2(A)$	1e-02	1e-02
4.27	$1/\kappa_2(A)$	3e-03	3e-03
	1	3e-03	3e-03
	$\kappa_2(A)$	3e-03	3e-03

Triw, $n = 1000$, $\kappa_2(A) = 5.33 \cdot 10^5$
($\alpha = -0.012$)

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	2e-07	2e-07
	1	5e-07	5e-07
	$\kappa_2(A)$	2e-07	2e-07
1	$1/\kappa_2(A)$	1e-02	1e-02
	1	2e-02	2e-02
	$\kappa_2(A)$	1e-02	1e-02
2.77	$1/\kappa_2(A)$	2e-03	2e-03
	1	4e-03	4e-03
	$\kappa_2(A)$	2e-03	2e-03

Triw, $n = 1000$, $\kappa_2(A) = 1.30 \cdot 10^7$
($\alpha = -0.015$)

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	3e-07	3e-07
	1	9e-08	9e-08
	$\kappa_2(A)$	3e-07	3e-07
1	$1/\kappa_2(A)$	2e-01	2e-01
	1	3e-01	3e-01
	$\kappa_2(A)$	2e-01	2e-01
3.11	$1/\kappa_2(A)$	3e-02	3e-02
	1	9e-02	9e-02
	$\kappa_2(A)$	8e-02	8e-02

Table 3.1

Backward Error η_2.

Hilbert, $n = 5$, $\kappa_2(A) = 4.77 \cdot 10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	3e-08	1e-07
	1	1e-08	2e-07
	$\kappa_2(A)$	3e-08	4e-08
1	$1/\kappa_2(A)$	2e-09	3e-08
	1	8e-09	2e-08
	$\kappa_2(A)$	4e-09	3e-08
11.1	$1/\kappa_2(A)$	6e-09	2e-08
	1	9e-09	4e-08
	$\kappa_2(A)$	7e-09	2e-08

Vandermonde, $n = 8$, $\kappa_2(A) = 2.68 \cdot 10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	5e-08	6e-08
	1	2e-08	4e-08
	$\kappa_2(A)$	6e-08	8e-08
1	$1/\kappa_2(A)$	5e-09	1e-08
	1	8e-09	2e-08
	$\kappa_2(A)$	7e-09	2e-08
4	$1/\kappa_2(A)$	3e-09	2e-08
	1	7e-09	2e-08
	$\kappa_2(A)$	2e-09	2e-08

Clement $n = 40$, $\kappa_2(A) = 2.11 \cdot 10^6$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	4e-08	3e-07
	1	3e-08	2e-07
	$\kappa_2(A)$	3e-08	3e-07
1	$1/\kappa_2(A)$	1e-08	2e-08
	1	2e-08	1e-08
	$\kappa_2(A)$	2e-08	2e-08
3.88	$1/\kappa_2(A)$	3e-08	3e-08
	1	2e-08	2e-08
	$\kappa_2(A)$	2e-08	2e-08

Dorr Matrix ($\theta = 0.1$), $n = 50$, $\kappa_2(A) = 1.33 \cdot 10^6$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	4e-08	3e-07
	1	4e-08	4e-07
	$\kappa_2(A)$	3e-08	3e-07
1	$1/\kappa_2(A)$	2e-08	4e-08
	1	2e-08	4e-08
	$\kappa_2(A)$	2e-08	5e-08
1.96	$1/\kappa_2(A)$	2e-08	5e-08
	1	2e-08	5e-08
	$\kappa_2(A)$	2e-08	5e-08

TABLE 3.2

Backward Error η_2, Continued.

Compan, $n = 1000$, $\kappa_2(A) = 3.35 \cdot 10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	2e-07	2e-07
	1	9e-07	1e-07
	$\kappa_2(A)$	2e-07	2e-07
1	$1/\kappa_2(A)$	8e-09	3e-09
	1	2e-08	1e-08
	$\kappa_2(A)$	2e-08	2e-08
1.31	$1/\kappa_2(A)$	1e-08	2e-08
	1	1e-08	9e-09
	$\kappa_2(A)$	2e-08	2e-08

Tridiag(-1,2,-1), $n = 1000$, $\kappa_2(A) = 4.06 \cdot 10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	3e-08	5e-07
	1	2e-08	6e-07
	$\kappa_2(A)$	3e-08	5e-07
1	$1/\kappa_2(A)$	2e-08	4e-08
	1	2e-08	5e-08
	$\kappa_2(A)$	2e-08	5e-08
1.28	$1/\kappa_2(A)$	2e-08	5e-08
	1	2e-08	4e-08
	$\kappa_2(A)$	2e-08	5e-08

Chow, $n = 1000$, $\kappa_2(A) = 5.42 \cdot 10^5$
($\alpha = 1.0$, $\delta = 0.5$)

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	6e-07	3e-05
	1	2e-07	3e-05
	$\kappa_2(A)$	3e-07	3e-05
1	$1/\kappa_2(A)$	3e-10	5e-07
	1	3e-10	5e-07
	$\kappa_2(A)$	3e-10	5e-07
99.4	$1/\kappa_2(A)$	6e-10	6e-07
	1	2e-10	6e-07
	$\kappa_2(A)$	3e-10	6e-07

Dorr, $n = 1000$, $\kappa_2(A) = 7.18 \cdot 10^5$
($\theta = 0.1$)

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	3e-08	4e-07
	1	2e-08	5e-07
	$\kappa_2(A)$	3e-08	4e-07
1	$1/\kappa_2(A)$	2e-08	4e-08
	1	2e-08	4e-08
	$\kappa_2(A)$	2e-08	5e-08
1.34	$1/\kappa_2(A)$	2e-08	4e-08
	1	2e-08	5e-08
	$\kappa_2(A)$	3e-08	4e-08

Fiedler, $n = 1000$, $\kappa_2(A) = 6.95 \cdot 10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	6e-07	4e-06
	1	1e-06	5e-06
	$\kappa_2(A)$	5e-07	4e-06
1	$1/\kappa_2(A)$	1e-07	5e-08
	1	1e-07	5e-08
	$\kappa_2(A)$	1e-07	5e-08
3.19	$1/\kappa_2(A)$	2e-08	5e-08
	1	2e-08	5e-08
	$\kappa_2(A)$	2e-08	6e-08

Minij, $n = 1000$, $\kappa_2(A) = 1.62 \cdot 10^6$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	6e-07	5e-05
	1	3e-07	5e-05
	$\kappa_2(A)$	2e-07	5e-05
1	$1/\kappa_2(A)$	1e-11	3e-07
	1	3e-10	3e-07
	$\kappa_2(A)$	3e-11	3e-07
3.19	$1/\kappa_2(A)$	1e-12	5e-07
	1	5e-11	5e-07
	$\kappa_2(A)$	2e-12	5e-07

TABLE 3.3

Backward Error η_2 for Triangular Matrices.

Kahan, $n = 40$, $\kappa_2(A) = 7.65 \cdot 10^6$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	8e-08	8e-08
	1	7e-08	7e-08
	$\kappa_2(A)$	9e-08	9e-08
1	$1/\kappa_2(A)$	9e-09	9e-09
	1	9e-09	9e-09
	$\kappa_2(A)$	5e-09	5e-09
2.68	$1/\kappa_2(A)$	7e-09	7e-09
	1	9e-09	9e-09
	$\kappa_2(A)$	8e-09	8e-09

R(Compan), $n = 1000$, $\kappa_2(A) = 3.35 \cdot 10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	1e-07	1e-07
	1	3e-07	3e-07
	$\kappa_2(A)$	7e-08	7e-08
1	$1/\kappa_2(A)$	6e-09	6e-09
	1	1e-08	1e-08
	$\kappa_2(A)$	1e-08	1e-08
16.7	$1/\kappa_2(A)$	4e-08	4e-08
	1	7e-08	7e-08
	$\kappa_2(A)$	1e-07	1e-07

R(Dorr) ($\theta = .01$), $n = 1000$, $\kappa_2(A) = 7.18 \cdot 10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	4e-08	4e-08
	1	3e-08	3e-08
	$\kappa_2(A)$	4e-08	4e-08
1	$1/\kappa_2(A)$	2e-08	2e-08
	1	2e-08	2e-08
	$\kappa_2(A)$	2e-08	2e-08
4.27	$1/\kappa_2(A)$	2e-08	2e-08
	1	2e-08	2e-08
	$\kappa_2(A)$	2e-08	2e-08

Triw, $n = 1000$, $\kappa_2(A) = 5.33 \cdot 10^5$
($\alpha = -0.012$)

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	2e-07	2e-07
	1	5e-07	5e-07
	$\kappa_2(A)$	2e-07	2e-07
1	$1/\kappa_2(A)$	2e-08	2e-08
	1	3e-08	3e-08
	$\kappa_2(A)$	2e-08	2e-08
2.77	$1/\kappa_2(A)$	9e-09	9e-09
	1	2e-08	2e-08
	$\kappa_2(A)$	8e-09	8e-09

Triw, $n = 1000$, $\kappa_2(A) = 1.30 \cdot 10^7$
($\alpha = -0.015$)

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	3e-07	3e-07
	1	9e-08	9e-08
	$\kappa_2(A)$	3e-07	3e-07
1	$1/\kappa_2(A)$	1e-08	1e-08
	1	2e-08	2e-08
	$\kappa_2(A)$	1e-08	1e-08
3.11	$1/\kappa_2(A)$	8e-09	8e-09
	1	2e-08	2e-08
	$\kappa_2(A)$	2e-08	2e-08

TABLE 4.1

Computable Backward Error $\hat{\eta}_2$.

Hilbert, $n = 5$, $\kappa_2(A) = 4.77 \cdot 10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	3e-08	1e-07
	1	1e-08	2e-07
	$\kappa_2(A)$	3e-08	4e-08
1	$1/\kappa_2(A)$	2e-09	3e-08
	1	8e-09	2e-08
	$\kappa_2(A)$	4e-09	2e-08
11.1	$1/\kappa_2(A)$	6e-09	2e-08
	1	9e-09	4e-08
	$\kappa_2(A)$	7e-09	2e-08

Vandermonde, $n = 8$, $\kappa_2(A) = 2.68 \cdot 10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	5e-08	6e-08
	1	2e-08	4e-08
	$\kappa_2(A)$	6e-08	8e-08
1	$1/\kappa_2(A)$	5e-09	1e-08
	1	8e-09	2e-08
	$\kappa_2(A)$	7e-09	2e-08
4	$1/\kappa_2(A)$	3e-09	2e-08
	1	7e-09	2e-08
	$\kappa_2(A)$	2e-09	2e-08

Clement, $n = 40$, $\kappa_2(A) = 2.11 \cdot 10^6$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	4e-08	3e-07
	1	3e-08	2e-07
	$\kappa_2(A)$	3e-08	3e-07
1	$1/\kappa_2(A)$	1e-08	2e-08
	1	2e-08	1e-08
	$\kappa_2(A)$	2e-08	2e-08
3.88	$1/\kappa_2(A)$	3e-08	3e-08
	1	2e-08	2e-08
	$\kappa_2(A)$	2e-08	2e-08

Dorr ($\theta = 0.01$), $n = 50$, $\kappa_2(A) = 1.33 \cdot 10^6$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	4e-08	3e-07
	1	4e-08	4e-07
	$\kappa_2(A)$	3e-08	3e-07
1	$1/\kappa_2(A)$	2e-08	5e-08
	1	2e-08	4e-08
	$\kappa_2(A)$	2e-08	5e-08
1.96	$1/\kappa_2(A)$	2e-08	4e-08
	1	2e-08	4e-08
	$\kappa_2(A)$	2e-08	5e-08

TABLE 4.2

Computable Backward Error $\hat{\eta}_2$, Continued.

Compan, $n = 1000$, $\kappa_2(A) = 3.35 \cdot 10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	2e-07	2e-07
	1	9e-07	1e-07
	$\kappa_2(A)$	2e-07	2e-07
1	$1/\kappa_2(A)$	8e-09	3e-09
	1	2e-08	1e-08
	$\kappa_2(A)$	2e-08	2e-08
1.31	$1/\kappa_2(A)$	1e-08	2e-08
	1	1e-08	9e-09
	$\kappa_2(A)$	2e-08	2e-08

Tridiag(-1,2-1), $n = 1000$, $\kappa_2(A) = 4.06 \cdot 10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	3e-08	5e-07
	1	2e-08	6e-07
	$\kappa_2(A)$	3e-08	5e-07
1	$1/\kappa_2(A)$	2e-08	4e-08
	1	2e-08	5e-08
	$\kappa_2(A)$	2e-08	5e-08
1.28	$1/\kappa_2(A)$	2e-08	5e-08
	1	2e-08	4e-08
	$\kappa_2(A)$	2e-08	5e-08

Chow, $n = 1000$, $\kappa_2(A) = 5.42 \cdot 10^5$
($\alpha = 1.0$, $\delta = 0.5$)

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	6e-07	3e-05
	1	2e-07	3e-05
	$\kappa_2(A)$	3e-07	3e-05
1	$1/\kappa_2(A)$	3e-10	5e-07
	1	3e-10	5e-07
	$\kappa_2(A)$	3e-10	5e-07
99.4	$1/\kappa_2(A)$	6e-10	6e-07
	1	2e-10	6e-07
	$\kappa_2(A)$	3e-10	6e-07

Dorr, $n = 1000$, $\kappa_2(A) = 7.18 \cdot 10^5$
($\theta = 0.1$)

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	3e-08	4e-07
	1	2e-08	5e-07
	$\kappa_2(A)$	3e-08	4e-07
1	$1/\kappa_2(A)$	2e-08	4e-08
	1	2e-08	4e-08
	$\kappa_2(A)$	2e-08	4e-08
1.34	$1/\kappa_2(A)$	2e-08	4e-08
	1	2e-08	5e-08
	$\kappa_2(A)$	3e-08	4e-08

Fiedler, $n = 1000$, $\kappa_2(A) = 6.95 \cdot 10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	6e-07	2e-06
	1	1e-06	2e-06
	$\kappa_2(A)$	5e-07	2e-06
1	$1/\kappa_2(A)$	1e-07	5e-08
	1	1e-07	5e-08
	$\kappa_2(A)$	1e-07	5e-08
3.19	$1/\kappa_2(A)$	2e-08	5e-08
	1	2e-08	5e-08
	$\kappa_2(A)$	2e-08	6e-08

Minij, $n = 1000$, $\kappa_2(A) = 1.62 \cdot 10^6$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	6e-07	8e-07
	1	3e-07	8e-07
	$\kappa_2(A)$	2e-07	8e-07
1	$1/\kappa_2(A)$	1e-11	2e-07
	1	3e-10	2e-07
	$\kappa_2(A)$	3e-11	2e-07
3.19	$1/\kappa_2(A)$	1e-12	5e-07
	1	5e-11	5e-07
	$\kappa_2(A)$	2e-12	5e-07

TABLE 4.3

Computable Backward Error $\hat{\eta}_2$ for Triangular Matrices.

Kahan, $n = 40$, $\kappa_2(A) = 7.65 \cdot 10^6$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	8e-08	8e-08
	1	7e-08	7e-08
	$\kappa_2(A)$	9e-08	9e-08
1	$1/\kappa_2(A)$	9e-09	9e-09
	1	9e-09	9e-09
	$\kappa_2(A)$	5e-09	5e-09
2.68	$1/\kappa_2(A)$	7e-09	7e-09
	1	9e-09	9e-09
	$\kappa_2(A)$	8e-09	8e-09

R(Compan), $n = 1000$, $\kappa_2(A) = 3.35 \cdot 10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	1e-07	1e-07
	1	3e-07	3e-07
	$\kappa_2(A)$	7e-08	7e-08
1	$1/\kappa_2(A)$	6e-09	6e-09
	1	1e-08	1e-08
	$\kappa_2(A)$	1e-08	1e-08
16.7	$1/\kappa_2(A)$	4e-08	4e-08
	1	7e-08	7e-08
	$\kappa_2(A)$	1e-07	1e-07

R(Dorr) ($\theta = .01$), $n = 1000$, $\kappa_2(A) = 7.18 \cdot 10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	4e-08	4e-08
	1	3e-08	3e-08
	$\kappa_2(A)$	4e-08	4e-08
1	$1/\kappa_2(A)$	2e-08	2e-08
	1	2e-08	2e-08
	$\kappa_2(A)$	2e-08	2e-08
4.27	$1/\kappa_2(A)$	2e-08	2e-08
	1	2e-08	2e-08
	$\kappa_2(A)$	2e-08	2e-08

Triw, $n = 1000$, $\kappa_2(A) = 5.33 \cdot 10^5$
($\alpha = -0.012$)

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	2e-07	2e-07
	1	5e-07	5e-07
	$\kappa_2(A)$	2e-07	2e-07
1	$1/\kappa_2(A)$	2e-08	2e-08
	1	3e-08	3e-08
	$\kappa_2(A)$	2e-08	2e-08
2.77	$1/\kappa_2(A)$	9e-09	9e-09
	1	2e-08	2e-08
	$\kappa_2(A)$	8e-09	8e-09

Triw, $n = 1000$, $\kappa_2(A) = 1.30 \cdot 10^7$
($\alpha = -0.015$)

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	3e-07	3e-07
	1	9e-08	9e-08
	$\kappa_2(A)$	3e-07	3e-07
1	$1/\kappa_2(A)$	1e-08	1e-08
	1	2e-08	2e-08
	$\kappa_2(A)$	1e-08	1e-08
3.11	$1/\kappa_2(A)$	8e-09	8e-09
	1	2e-08	2e-08
	$\kappa_2(A)$	2e-08	2e-08

TABLE 8.1

Relative Error $\|x-\hat{x}\|_2/\|x\|_2$ *and Upper Bound* $\kappa_2(A)\eta_2$.

Hilbert, $n=5$, $\kappa_2(A)=4.77\cdot 10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR	$\kappa_2(A)\eta_2$ GE	$\kappa_2(A)\eta_2$ QR
$\kappa_2(A)$	$1/\kappa_2(A)$	4e-03	2e-03	1e-02	5e-02
	1	2e-03	1e-03	6e-03	1e-01
	$\kappa_2(A)$	2e-03	5e-03	1e-02	2e-02
1	$1/\kappa_2(A)$	7e-04	6e-04	1e-03	1e-02
	1	7e-04	6e-04	4e-03	9e-03
	$\kappa_2(A)$	7e-04	6e-04	2e-03	1e-02
11.1	$1/\kappa_2(A)$	7e-04	6e-04	3e-03	8e-03
	1	7e-04	6e-04	4e-03	2e-02
	$\kappa_2(A)$	7e-04	6e-04	3e-03	8e-03

Vandermonde, $n=8$, $\kappa_2(A)=2.68\cdot 10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR	$\kappa_2(A)\eta_2$ GE	$\kappa_2(A)\eta_2$ QR
$\kappa_2(A)$	$1/\kappa_2(A)$	3e-03	5e-03	1e-02	2e-02
	1	2e-03	6e-03	6e-03	1e-02
	$\kappa_2(A)$	1e-03	6e-03	2e-02	2e-02
1	$1/\kappa_2(A)$	8e-05	6e-05	1e-03	4e-03
	1	8e-05	6e-05	2e-03	5e-03
	$\kappa_2(A)$	8e-05	6e-05	2e-03	4e-03
4	$1/\kappa_2(A)$	9e-05	6e-05	8e-04	5e-03
	1	9e-05	6e-05	2e-03	5e-03
	$\kappa_2(A)$	9e-05	6e-05	5e-04	5e-03

Clement, $n=40$, $\kappa_2(A)=2.11\cdot 10^6$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR	$\kappa_2(A)\eta_2$ GE	$\kappa_2(A)\eta_2$ QR
$\kappa_2(A)$	$1/\kappa_2(A)$	3e-04	1e-01	8e-02	6e-01
	1	1e-04	8e-02	5e-02	5e-01
	$\kappa_2(A)$	1e-06	2e-02	7e-02	6e-01
1	$1/\kappa_2(A)$	3e-08	6e-04	2e-02	4e-02
	1	1e-07	6e-04	5e-02	3e-02
	$\kappa_2(A)$	2e-07	6e-04	5e-02	5e-02
3.88	$1/\kappa_2(A)$	2e-07	4e-05	5e-02	5e-02
	1	1e-07	3e-05	3e-02	3e-02
	$\kappa_2(A)$	1e-07	4e-05	5e-02	4e-02

Dorr ($\theta=0.01$), $n=50$, $\kappa_2(A)=1.33\cdot 10^6$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR	$\kappa_2(A)\eta_2$ GE	$\kappa_2(A)\eta_2$ QR
$\kappa_2(A)$	$1/\kappa_2(A)$	1e-05	7e-03	6e-02	4e-01
	1	8e-05	3e-02	5e-02	5e-01
	$\kappa_2(A)$	6e-05	3e-02	4e-02	3e-01
1	$1/\kappa_2(A)$	7e-04	2e-02	3e-02	6e-02
	1	7e-04	2e-02	3e-02	6e-02
	$\kappa_2(A)$	7e-04	2e-02	2e-02	6e-02
1.96	$1/\kappa_2(A)$	7e-04	2e-02	2e-02	6e-02
	1	7e-04	2e-02	2e-02	6e-02
	$\kappa_2(A)$	7e-04	2e-02	2e-02	6e-02

Table 8.2

Relative Error $\|x - \hat{x}\|_2/\|x\|_2$ and Upper Bound $\kappa_2(A)\eta_2$, Continued.

Compan, $n = 1000$, $\kappa_2(A) = 3.35 \cdot 10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR	$\kappa_2(A)\eta_2$	
				GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	2e-03	4e-03	8e-02	7e-02
	1	1e-02	1e-03	3e-01	5e-02
	$\kappa_2(A)$	4e-03	3e-03	7e-02	8e-02
1	$1/\kappa_2(A)$	1e-04	6e-05	3e-03	1e-03
	1	3e-04	2e-04	8e-03	4e-03
	$\kappa_2(A)$	2e-04	2e-04	6e-03	5e-03
1.31	$1/\kappa_2(A)$	2e-04	1e-04	5e-03	6e-03
	1	2e-04	2e-04	3e-03	3e-03
	$\kappa_2(A)$	3e-04	2e-04	8e-03	6e-03

Tridiag(-1,2,-1), $n = 1000$, $\kappa_2(A) = 4.06 \cdot 10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR	$\kappa_2(A)\eta_2$	
				GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	2e-04	4e-02	1e-02	2e-01
	1	7e-05	8e-02	9e-03	2e-01
	$\kappa_2(A)$	9e-05	6e-03	1e-02	2e-01
1	$1/\kappa_2(A)$	1e-04	9e-03	8e-03	2e-02
	1	1e-04	9e-03	8e-03	2e-02
	$\kappa_2(A)$	1e-04	9e-03	7e-03	2e-02
1.28	$1/\kappa_2(A)$	1e-04	9e-03	8e-03	2e-02
	1	1e-04	9e-03	7e-03	2e-02
	$\kappa_2(A)$	1e-04	9e-03	8e-03	2e-02

Chow, $n = 1000$, $\kappa_2(A) = 5.42 \cdot 10^5$
($\alpha = 1.0$, $\delta = 0.5$)

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR	$\kappa_2(A)\eta_2$	
				GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	1e-03	6e-02	3e-01	2e+01
	1	6e-04	6e-02	1e-01	2e+01
	$\kappa_2(A)$	3e-04	6e-02	1e-01	2e+01
1	$1/\kappa_2(A)$	9e-07	5e-03	1e-04	3e-01
	1	2e-07	5e-03	1e-04	3e-01
	$\kappa_2(A)$	2e-07	5e-03	1e-04	3e-01
99.4	$1/\kappa_2(A)$	9e-07	5e-03	3e-04	3e-01
	1	4e-07	5e-03	1e-04	3e-01
	$\kappa_2(A)$	3e-07	5e-03	2e-04	3e-01

Dorr, $n = 1000$, $\kappa_2(A) = 7.18 \cdot 10^5$
($\theta = 0.1$)

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR	$\kappa_2(A)\eta_2$	
				GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	2e-05	1e-03	2e-02	3e-01
	1	7e-05	6e-03	2e-02	4e-01
	$\kappa_2(A)$	4e-05	6e-03	2e-02	3e-01
1	$1/\kappa_2(A)$	2e-04	1e-02	1e-02	3e-02
	1	3e-04	1e-02	1e-02	3e-02
	$\kappa_2(A)$	5e-04	9e-03	2e-02	3e-02
1.34	$1/\kappa_2(A)$	2e-04	1e-02	1e-02	3e-02
	1	4e-04	1e-02	1e-02	3e-02
	$\kappa_2(A)$	4e-04	1e-02	2e-02	3e-02

Fiedler, $n = 1000$, $\kappa_2(A) = 6.95 \cdot 10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR	$\kappa_2(A)\eta_2$	
				GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	2e-01	2e+00	4e-01	3e+00
	1	2e-01	3e+00	7e-01	4e+00
	$\kappa_2(A)$	2e-01	2e+00	4e-01	3e+00
1	$1/\kappa_2(A)$	5e-02	2e-02	8e-02	3e-02
	1	5e-02	2e-02	8e-02	3e-02
	$\kappa_2(A)$	5e-02	2e-02	8e-02	3e-02
3.19	$1/\kappa_2(A)$	6e-03	2e-02	1e-02	4e-02
	1	6e-03	2e-02	1e-02	4e-02
	$\kappa_2(A)$	6e-03	2e-02	1e-02	4e-02

Minij, $n = 1000$, $\kappa_2(A) = 1.62 \cdot 10^6$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR	$\kappa_2(A)\eta_2$	
				GE	QR
$\kappa_2(A)$	$1/\kappa_2(A)$	3e-02	6e+01	9e-01	8e+01
	1	2e-02	6e+01	4e-01	8e+01
	$\kappa_2(A)$	3e-02	6e+01	4e-01	8e+01
1	$1/\kappa_2(A)$	3e-07	3e-01	2e-05	4e-01
	1	4e-07	3e-01	4e-04	4e-01
	$\kappa_2(A)$	2e-07	3e-01	6e-05	4e-01
3.19	$1/\kappa_2(A)$	1e-07	4e-01	2e-06	9e-01
	1	2e-07	4e-01	8e-05	9e-01
	$\kappa_2(A)$	1e-07	4e-01	4e-06	9e-01

TABLE 8.3

Relative Error $\|x-\hat{x}\|_2/\|x\|_2$ and Upper Bound $\kappa_2(A)\eta_2$ for Triangular Matrices.

Kahan, $n=40$, $\kappa_2(A)=7.65\cdot10^6$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR	$\kappa_2(A)\eta_2$ GE	$\kappa_2(A)\eta_2$ QR
$\kappa_2(A)$	$1/\kappa_2(A)$	1e-03	1e-03	6e-01	6e-01
	1	2e-04	2e-04	5e-01	5e-01
	$\kappa_2(A)$	3e-04	3e-04	7e-01	7e-01
1	$1/\kappa_2(A)$	7e-08	7e-08	7e-02	7e-02
	1	1e-07	1e-07	7e-02	7e-02
	$\kappa_2(A)$	7e-08	7e-08	4e-02	4e-02
2.68	$1/\kappa_2(A)$	7e-08	7e-08	5e-02	5e-02
	1	1e-07	1e-07	7e-02	7e-02
	$\kappa_2(A)$	3e-08	3e-08	6e-02	6e-02

R(Compan), $n=1000$, $\kappa_2(A)=3.35\cdot10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR	$\kappa_2(A)\eta_2$ GE	$\kappa_2(A)\eta_2$ QR
$\kappa_2(A)$	$1/\kappa_2(A)$	3e-04	3e-04	1e-01	1e-01
	1	6e-04	6e-04	3e-01	3e-01
	$\kappa_2(A)$	7e-05	7e-05	1e-01	1e-01
1	$1/\kappa_2(A)$	9e-07	9e-07	1e-02	1e-02
	1	5e-07	5e-07	2e-02	2e-02
	$\kappa_2(A)$	4e-07	4e-07	1e-02	1e-02
1.31	$1/\kappa_2(A)$	1e-06	1e-06	5e-03	5e-03
	1	2e-07	2e-07	1e-02	1e-02
	$\kappa_2(A)$	1e-06	1e-06	4e-03	4e-03

R(Dorr) ($\theta=.01$), $n=1000$, $\kappa_2(A)=7.18\cdot10^5$

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR	$\kappa_2(A)\eta_2$ GE	$\kappa_2(A)\eta_2$ QR
$\kappa_2(A)$	$1/\kappa_2(A)$	1e-03	1e-03	3e-02	3e-02
	1	4e-04	4e-04	2e-02	2e-02
	$\kappa_2(A)$	2e-04	2e-04	3e-02	3e-02
1	$1/\kappa_2(A)$	2e-04	2e-04	1e-02	1e-02
	1	4e-05	4e-05	1e-02	1e-02
	$\kappa_2(A)$	1e-04	1e-04	1e-02	1e-02
1.31	$1/\kappa_2(A)$	3e-05	3e-05	1e-02	1e-02
	1	9e-05	9e-05	1e-02	1e-02
	$\kappa_2(A)$	1e-04	1e-04	1e-02	1e-02

Triw, $n=1000$, $\kappa_2(A)=5.33\cdot10^5$
($\alpha=-0.012$)

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR	$\kappa_2(A)\eta_2$ GE	$\kappa_2(A)\eta_2$ QR
$\kappa_2(A)$	$1/\kappa_2(A)$	3e-04	3e-04	1e-01	1e-01
	1	6e-04	6e-04	3e-01	3e-01
	$\kappa_2(A)$	7e-05	7e-05	1e-01	1e-01
1	$1/\kappa_2(A)$	9e-07	9e-07	1e-02	1e-02
	1	5e-07	5e-07	2e-02	2e-02
	$\kappa_2(A)$	4e-07	4e-07	1e-02	1e-02
2.77	$1/\kappa_2(A)$	1e-06	1e-06	5e-03	5e-03
	1	2e-07	2e-07	1e-02	1e-02
	$\kappa_2(A)$	1e-06	1e-06	4e-03	4e-03

Triw, $n=1000$, $\kappa_2(A)=1.30\cdot10^7$
($\alpha=-0.015$)

$\kappa_2(A,b)$	$\|b\|_2$	GE	QR	$\kappa_2(A)\eta_2$ GE	$\kappa_2(A)\eta_2$ QR
$\kappa_2(A)$	$1/\kappa_2(A)$	7e-03	7e-03	4e+00	4e+00
	1	7e-02	7e-02	1e+00	1e+00
	$\kappa_2(A)$	9e-02	9e-02	4e+00	4e+00
1	$1/\kappa_2(A)$	3e-07	3e-07	2e-01	2e-01
	1	2e-07	2e-07	3e-01	3e-01
	$\kappa_2(A)$	1e-06	1e-06	2e-01	2e-01
3.11	$1/\kappa_2(A)$	1e-06	1e-06	1e-01	1e-01
	1	8e-07	8e-07	3e-01	3e-01
	$\kappa_2(A)$	4e-07	4e-07	3e-01	3e-01