# THREE ABSOLUTE PERTURBATION BOUNDS FOR MATRIX EIGENVALUES IMPLY RELATIVE BOUNDS

STANLEY C. EISENSTAT* AND ILSE C. F. IPSEN†

**Abstract.** We show that three well-known perturbation bounds for matrix eigenvalues imply relative bounds: the Bauer-Fike and Hoffman-Wielandt theorems for diagonalisable matrices, and Weyl's theorem for Hermitian matrices. As a consequence, relative perturbation bounds are not necessarily stronger than absolute bounds; and the conditioning of an eigenvalue in the relative sense is the same as in the absolute sense.

We also show that eigenvalues of normal matrices are no more sensitive to perturbations than eigenvalues of Hermitian positive-definite matrices. The relative error bounds are invariant under congruence transformations, such as grading and scaling.

## 1. Introduction.

Let $A$ be a complex square matrix; and let $A+E$ be a perturbation of $A$. We want to estimate the error in an eigenvalue $\hat{\lambda}$ of $A+E$ when it is viewed as an approximation to an eigenvalue of $A$.

Traditional perturbation results bound the absolute error in an eigenvalue. The Bauer-Fike theorem [2, Theorem IIIa], for instance, bounds the absolute distance between $\hat{\lambda}$ and a closest eigenvalue $\lambda$ of a diagonalisable matrix $A$ by

$$|\lambda - \hat{\lambda}| \leq \kappa(X)\,\|E\|,$$

where $\kappa(X) = \|X\|\,\|X^{-1}\|$ is the condition number of an eigenvector matrix $X$ of $A$.

The simplest way to generate a relative perturbation bound is to divide an absolute error bound by a (non-zero) eigenvalue. In case of the Bauer-Fike theorem we get

$$\frac{|\lambda - \hat{\lambda}|}{|\lambda|} \leq \kappa(X)\,\frac{\|E\|}{|\lambda|}.$$

Now the bound depends on $\lambda$. In particular, the bound is smaller for eigenvalues $\lambda$ that are large in magnitude than for those that are small in magnitude.

However this kind of relative perturbation bound may not be good enough, because there are algorithms that compute all eigenvalues to high relative accuracy, even those of small magnitude. Among such algorithms are Jacobi methods for Hermitian positive-definite matrices [4, 13], and the dqds algorithm for certain tridiagonal matrices [7]. These algorithms have 'genuine' relative error bounds that do not depend on the eigenvalues.

Our original motivation was to determine under which circumstances we can find genuine relative perturbation bounds that do not depend on the eigenvalues. In particular, does the existence of such a bound depend on the properties of the matrix (e.g. Hermitian positive-definite) or on the properties of the perturbation (e.g. relative component-wise)?

Our answer is that genuine relative perturbation bounds exist whenever absolute bounds exist, for almost any matrix and for any perturbation. In particular we show

* Department of Computer Science, Yale University, P. O. Box 208285, New Haven, CT 06520-8285 (eisenstat-stan@cs.yale.edu). The research of this author was supported in part by NSF grant CCR-9400921.

† Center for Research In Scientific Computation, Department of Mathematics, North Carolina State University, P. O. Box 8205, Raleigh, NC 27695-8205 (ipsen@math.ncsu.edu). The research of this author was supported in part by NSF grant CCR-9400921.

that three well-known absolute bounds imply genuine relative bounds. In this sense relative bounds are no stronger than absolute bounds. We also show that corresponding absolute and relative perturbation bounds have the same condition number.

**1.1. Overview.** In §2 we show that the Bauer-Fike Theorem for diagonalisable matrices implies a large class of relative bounds. The condition number is the same for relative and absolute bounds. We conclude that the eigenvalues of a normal, non-singular matrix are well-conditioned, in the absolute as well as in the relative sense.

In §3 we derive a relative perturbation bound for normal matrices that suggests that the eigenvalues of a normal matrix are as well conditioned as the eigenvalues of its positive-definite polar factor. The bound is invariant under congruence transformations. Hence the eigenvalues of a graded, normal matrix are no more sensitive to perturbations than the eigenvalues of an 'ungraded' Hermitian positive-definite matrix.

In §4 we show that Weyl's perturbation theorem implies a relative bound that is slightly stronger than existing bounds.

In §5 we extend the Hoffman-Wielandt theorem for diagonalisable matrices and show that it implies a relative bound.

**1.2. Notation.** $I$ is the identity matrix; $\|\cdot\|$ is the two-norm and $\|\cdot\|_F$ the Frobenius norm; $A^*$ is the conjugate transpose of a matrix $A$; and $\kappa(X) \equiv \|X\|\,\|X^{-1}\|$ is the two norm condition number of a matrix $X$ with respect to inversion.

**2. Two-norm bounds for diagonalisable matrices.** We show that the Bauer-Fike Theorem implies a relative bound.

Let $A$ be a diagonalisable matrix with eigendecomposition $A = X\Lambda X^{-1}$, where

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

and $\lambda_i$ are the eigenvalues of $A$. The Bauer-Fike Theorem bounds the absolute error between a perturbed eigenvalue and a closest eigenvalue of $A$.

THEOREM 2.1 (THEOREM IIIA IN [2]). *If $A$ is diagonalisable then*

$$\min_i |\lambda_i - \hat\lambda| \le \kappa(X)\,\|E\|,$$

*where* $\kappa(X) \equiv \|X\|\,\|X^{-1}\|$.

The Bauer-Fike Theorem implies the relative bound below, provided $A$ is non-singular.

COROLLARY 2.2. *If $A$ is diagonalisable and non-singular, then*

$$\min_i \frac{|\lambda_i - \hat\lambda|}{|\lambda_i|} \le \kappa(X)\,\|A^{-1}E\|.$$

*Proof.* Write $(A+E)\hat x = \hat\lambda \hat x$ as

$$(\bar A + \bar E)\,\hat x = \hat x, \qquad \text{where} \qquad \bar A \equiv \hat\lambda A^{-1}, \quad \bar E \equiv -A^{-1}E.$$

Hence 1 is an eigenvalue of $\bar A + \bar E$. The matrix $\bar A$ has eigenvalues $\hat\lambda/\lambda_i$ and the same eigenvector matrix as $A$. Apply Theorem 2.1 to $\bar A$ and the eigenvalue 1 of $\bar A + \bar E$. □

We interpret the amplifier $\kappa(X)$ in the bounds as a condition number for the eigenvalues of $A$. Both absolute and relative perturbation bounds have the same condition number. The condition number indicates the sensitivity of an eigenvalue to absolute perturbations $E$ and to relative perturbations

$$A^{-1}E = A^{-1}\left((A+E)-A\right).$$

Note however that an eigenvalue closest to $\hat{\lambda}$ in the absolute sense may be different from an eigenvalue closest to $\hat{\lambda}$ in the relative sense. Corollary 2.2 generalises [16, Theorem 3.12].

**2.1. A larger class of relative perturbations.** So far we have expressed relative perturbations as $A^{-1}E$. But why confine $A^{-1}$ to the left of $E$? Why not move it to the right? Or why not factor $A^{-1}$ and distribute the factors on both sides of $E$? The bound below is a consequence of Corollary 2.2.

THEOREM 2.3. *Let $A$ be diagonalisable and non-singular. If $A = A_1 A_2$ then*

$$\min_i \frac{|\lambda_i - \hat{\lambda}|}{|\lambda_i|} \leq \kappa(W)\, \|A_1^{-1} E A_2^{-1}\|,$$

*where $W$ is an eigenvector matrix of $A_2 A_1$.*

*Proof.* Define

$$\bar{A} \equiv A_2 A A_2^{-1}, \qquad \bar{E} \equiv A_2 E A_2^{-1}.$$

Since $\bar{A}$ is similar to $A$, it is diagonalisable with eigendecomposition $\bar{A} = W \Lambda W^{-1}$. Applying Corollary 2.2 to $\bar{A}$ and $\bar{A} + \bar{E}$ gives

$$\min_i \frac{|\lambda_i - \hat{\lambda}|}{|\lambda_i|} \leq \kappa(W)\, \|\bar{A}^{-1}\bar{E}\| = \kappa(W)\, \|A_1^{-1} E A_2^{-1}\|.$$

□

When $A_1$ and $A_2$ commute, the original condition number $\kappa(X)$ returns.

COROLLARY 2.4. *Let $A = A_1 A_2$ be diagonalisable and non-singular. If $A_1 A_2 = A_2 A_1$ then*

$$\min_i \frac{|\lambda_i - \hat{\lambda}|}{|\lambda_i|} \leq \kappa(X)\, \|A_1^{-1} E A_2^{-1}\|.$$

When $A_1 = A$ and $A_2 = I$, we recover Corollary 2.2. The choice $A_1 = I$ and $A_2 = A$ gives a similar bound.

COROLLARY 2.5. *Let $A$ be diagonalisable and non-singular. Then*

$$\min_i \frac{|\lambda_i - \hat{\lambda}|}{|\lambda_i|} \leq \kappa(X)\, \|E A^{-1}\|.$$

Another popular choice for $A_1$ and $A_2$ is a square root $A^{1/2}$ of $A$. A square root of a matrix [9, p 54, Problem 7], [10, p 467, p 471] exists whenever the matrix is non-singular [10, p 468].

COROLLARY 2.6. *Let $A$ be diagonalisable and non-singular. Then*

$$\min_i \frac{|\lambda_i - \hat{\lambda}|}{|\lambda_i|} \leq \kappa(X)\, \|A^{-1/2} E A^{-1/2}\|.$$

**3. Two-norm bounds for normal matrices.** We derive relative perturbation bounds for normal matrices that are invariant under congruence transformations such as grading and scaling.

When applied to normal matrices, Corollary 2.4 simplifies.

COROLLARY 3.1. *Let $A = A_1 A_2$ be normal and non-singular. If $A_1 A_2 = A_2 A_1$ then*

$$\min_i \frac{|\lambda_i - \hat{\lambda}|}{|\lambda_i|} \leq \|A_1^{-1} E A_2^{-1}\|.$$

Therefore eigenvalues of normal matrices are well-conditioned, in the absolute sense as well as in many relative senses.

Up to now we have chosen the following factorisations for $A$,

$$(A_1, A_2) = (A, I), \qquad (A_1, A_2) = (I, A), \qquad (A_1, A_2) = (A^{1/2}, A^{1/2}).$$

In each case $A_1$ and $A_2$ commute and we retain the perfect conditioning of normal matrices. Normal matrices, however, admit another commuting factorisation. It results from the polar factorisation and the fact that polar factors of normal matrices commute.

Every non-singular matrix $A$ has a polar factorisation $A = HU$, where $H \equiv (AA^*)^{1/2}$ is Hermitian positive-definite, and $U \equiv H^{-1}A$ is unitary [9, Corollary 7.3.3]. We use the following property of polar factors.

LEMMA 3.2. *Let $A$ be normal and non-singular with polar factorisation $A = HU$. Then*

$$HU = UH = H^{1/2} U H^{1/2}.$$

*Proof.* The first equality follows from the fact that polar factors of a normal matrix commute [9, Theorem 7.3.4].

To prove the second equality, note that a Hermitian positive-definite matrix $H$ has a unique Hermitian positive-definite square-root $H^{1/2}$ [9, Theorem 7.2.6]. From the first equality $HU = UH$ follows

$$H = UHU^* = (UH^{1/2}U^*)\,(UH^{1/2}U^*).$$

Thus $UH^{1/2}U^*$ is also a Hermitian positive-definite square-root of $H$. But uniqueness implies $H^{1/2} = UH^{1/2}U^*$. This means $H^{1/2}U = UH^{1/2}$ and

$$A = HU = H^{1/2}\,(H^{1/2}U) = H^{1/2}UH^{1/2}.$$

□

The following bound is a consequence of Corollary 2.4.

THEOREM 3.3. *Let $A$ be normal and non-singular, with Hermitian positive-definite polar factor $H$. Then*

$$\min_i \frac{|\lambda_i - \hat{\lambda}|}{|\lambda_i|} \leq \|H^{-1/2} E H^{-1/2}\|.$$

*Proof.* $A = HU$ is a polar factorisation of $A$, and Lemma 3.2 implies $A = H^{1/2}UH^{1/2}$. Set

$$A_1 \equiv H^{1/2}U, \qquad A_2 \equiv H^{1/2}.$$

Then

$$\|A_1^{-1}EA_2^{-1}\| = \|U^*H^{-1/2}EH^{-1/2}\| = \|H^{-1/2}EH^{-1/2}\|.$$

Since $A_2A_1 = A_1A_2$, we can apply Corollary 2.4 to get the desired bound. □

Therefore the eigenvalues of a normal matrix have the same relative error bound as the eigenvalues of its positive-definite polar factor, which suggests that they are as well conditioned as the eigenvalues of its positive-definite polar factor.

**3.1. Grading.** The advantage of Theorem 3.3 is that it is invariant under congruence transformations in the following sense.

COROLLARY 3.4. *Let $A$ be normal and non-singular, with Hermitian positive-definite polar factor $H$. If $D$ is non-singular and*

$$E = DE_1D^*, \qquad H = DM_1D^*$$

*then*

$$\min_i \frac{|\lambda_i - \hat{\lambda}|}{|\lambda_i|} \leq \|M_1^{-1/2}E_1M_1^{-1/2}\|.$$

*Proof.* Theorem 3.3 implies

$$\min_i \frac{|\lambda_i - \hat{\lambda}|}{|\lambda_i|} \leq \|H^{-1/2}EH^{-1/2}\|.$$

Since $H$ is Hermitian positive-definite $M_1 = D^{-1}HD^{-*}$ is also Hermitian positive-definite and has a Hermitian square-root. Since

$$H = H^{1/2}H^{1/2} = DM_1^{1/2}\,(DM_1^{1/2})^*$$

are both 'Cholesky factorisations' of $H$, they are related by a unitary matrix $Q$, i.e. $H^{1/2} = (DM_1^{1/2})\,Q$. Hence, by Theorem 3.3,

$$\min_i \frac{|\lambda_i - \hat{\lambda}|}{|\lambda_i|} \leq \|H^{-1/2}EH^{-1/2}\| = \|M_1^{-1/2}E_1M_1^{-1/2}\|.$$

□

Therefore the relative error bound is invariant under congruence transformations extracted from the perturbation and the positive-definite polar factor. Corollary 3.4 implies essentially that the eigenvalues of a graded, normal matrix are no more sensitive than the eigenvalues of the best 'ungraded' positive-definite polar factor.

**3.2. Relation to existing work.** Slapničar and Veselić [15, §2], [16, §2] have obtained similar results. Their results are more general in the sense that they apply to the generalised eigenvalue problem $Ax = \lambda Bx$, where $A$ is Hermitian and $B$ is Hermitian positive-definite; and they bound the distance between the $i$th perturbed and exact eigenvalues. When $B = I$ their *absolute value of* $A$, $|A|$, turns out to be a

Hermitian positive-definite polar factor of $A$. However, our results are more general in the sense that they apply to a larger class of matrices (normal as opposed to Hermitian), to a larger class of perturbations (norm-wise as opposed to component-wise), and to a larger class of grading matrices (non-singular as opposed to real diagonal).

To relate our results to those in [15, §2], [16, §2], we assume that the backward error is scaled in the same way as the matrix so that $\|E_1\| \leq \epsilon\|M\|$, where $\epsilon$ is a small positive number and $A = DMD^*$. The following result is similar in spirit to [16, Theorem 2.13].

COROLLARY 3.5. *Let $A$ be normal and non-singular, with positive-definite polar factor $H$. Let $D$ be non-singular and*

$$A = DMD^*, \qquad E = DE_1D^*, \qquad H = DM_1D^*.$$

*If $\|E_1\| \leq \epsilon\|M\|$ then*

$$\min_i \frac{|\lambda_i - \hat{\lambda}|}{|\lambda_i|} \leq \epsilon\, \|M\|\|M_1^{-1}\|.$$

*Proof.* Corollary 3.4 implies

$$\min_i \frac{|\lambda_i - \hat{\lambda}|}{|\lambda_i|} \leq \|M_1^{-1/2}E_1M_1^{-1/2}\| \leq \epsilon\, \|M_1^{-1/2}\|^2\|M\|.$$

As a square root of the Hermitian positive-definite matrix $M_1$, $M_1^{1/2}$ is Hermitian. Therefore $\|M_1^{-1/2}\|^2 = \|M_1^{-1}\|$. □

Therefore the eigenvalues have small relative error when $M$ and $M_1^{-1}$ have small norm. Here, $M$ is what is left over after extracting the grading from $A$, and $M_1$ is what is left over after extracting the grading from the positive-definite polar factor. One might argue that in Corollary 3.5 the polar factor of $M$ would be preferable to the polar factor of $A$. But then we would be comparing apples and oranges. Because $A$ and $H$ have the same eigenvalues (in magnitude), we have to compare the scaled version of $A$ (which is $M$) to the scaled version of $H$ (which is $M_1$).

In the special case when $M$ is unitary we arrive at the same conclusion as [16, Theorem 2.37], namely that the eigenvalues of $A$ are well-conditioned.

COROLLARY 3.6. *If, in addition to the assumptions of Corollary 3.5, $D$ also commutes with the unitary polar factor of $A$ and $M$ is unitary, then*

$$\min_i \frac{|\lambda_i - \hat{\lambda}|}{|\lambda_i|} \leq \epsilon.$$

*Proof.* $\|M\| = 1$ because $M$ is unitary; and

$$A = UH = U\, DM_1D^* = D\, UM_1\, D^*$$

because $D$ and $U$ commute. But $A = DMD^*$ and the non-singularity of $D$ imply $M = UM_1$. Hence $M_1$ is unitary and $\|M_1^{-1}\| = 1$. □

In conclusion, our results only bound the error in a single eigenvalue of $A + E$, while other results bound the relative error in all eigenvalues of $A + E$ simultaneously. But this comes at a price. For instance, existing bounds for Hermitian matrices restrict the perturbation $E$ so that $A+E$ is Hermitian and has the same inertia as $A$ [1, Lemma 1], [16, Theorem 2.1]. Or they restrict the congruence transformation $D$ and the size of the perturbation [8, Corollary 5]. Hence we have traded simultaneous bounds for all eigenvalues against freedom in perturbations and applicability to a larger class of matrices.

**4. Two-norm bounds for Hermitian matrices.** We show that Weyl's Theorem implies a relative bound.

Let $A$ and $A+E$ be Hermitian with respective eigenvalues

$$\lambda_n \leq \ldots \leq \lambda_1, \qquad \hat{\lambda}_n \leq \ldots \leq \hat{\lambda}_1.$$

Weyl's Perturbation Theorem bounds the worst-case absolute error between the $i$th exact and perturbed eigenvalues of Hermitian matrices in terms of the two-norm.

THEOREM 4.1 (COROLLARY III.2.6 IN [3]). *If $A$ and $A+E$ are Hermitian then*

$$\max_{1\leq i\leq n} |\lambda_i - \hat{\lambda}_i| \leq \|E\|.$$

The absolute bound in Theorem 4.1 implies a relative bound, provided that the matrices are in addition positive-definite.

COROLLARY 4.2 (THEOREM 2.3 IN [14]). *If $A$ and $A+E$ are Hermitian positive-definite then*

$$\max_{1\leq i\leq n} \frac{|\lambda_i - \hat{\lambda}_i|}{|\lambda_i|} \leq \|A^{-1/2} E A^{-1/2}\|.$$

*Proof.* Fix an index $i$. Let $\hat{x}$ be an eigenvector of $A+E$ associated with $\hat{\lambda}_i$, i.e.

$$(A+E)\hat{x} = \hat{\lambda}_i \hat{x}.$$

Multiplying $(\hat{\lambda}_i I - E)\hat{x} = A\hat{x}$ by $A^{-1}$ gives

$$(\bar{A}+\bar{E})\, z = z, \qquad \text{where} \qquad \bar{A} \equiv \hat{\lambda}_i A^{-1}, \quad \bar{E} \equiv -A^{-1/2} E A^{-1/2}, \quad z \equiv A^{1/2}\hat{x}.$$

Hence 1 is an eigenvalue of $\bar{A}+\bar{E}$.

We will show that it is actually the $(n-i+1)$st eigenvalue. We argue as in the proof of [5, Theorem 2.1]. Since $\hat{\lambda}_i$ is the $i$th eigenvalue of $A+E$, 0 must be the $i$th eigenvalue of

$$(A+E) - \hat{\lambda}_i I = A^{1/2}\,(I - \bar{A} - \bar{E})\, A^{1/2}.$$

But this is a congruence transformation because square-roots of positive-definite matrices are Hermitian. Congruence transformations preserve the inertia. Hence 0 is the $i$th eigenvalue of $I - \bar{A} - \bar{E}$, and 1 is the $(n-i+1)$st eigenvalue of $\bar{A}+\bar{E}$.

Since $A+E$ is positive-definite, $\hat{\lambda}_i$ is positive and $\hat{\lambda}_i/\lambda_{n-j+1}$ is the $j$th eigenvalue of $\bar{A}$. Applying Theorem 4.1 to $\bar{A}$ and $\bar{A}+\bar{E}$ gives

$$\max_{1\leq j\leq n} \left| \frac{\hat{\lambda}_i}{\lambda_{n-j+1}} - \mu_j \right| \leq \|\bar{E}\| \leq \|A^{-1/2} E A^{-1/2}\|,$$

where $\mu_j$ are the eigenvalues of $\bar{A}+\bar{E}$. When $j = n-i+1$, then $\mu_{n-i+1} = 1$ and we get the desired bound. □

A slightly weaker bound with a restriction on $A+E$ appears in [11, Theorem 3.2].

**5. Frobenius norm bounds for diagonalisable matrices.** We show that a slightly stronger version of the Hoffman-Wielandt Theorem for diagonalisable matrices implies a relative bound. The idea for this proof was inspired by the derivation of relative error bounds for multiplicative perturbations in [11].

Let $A$ and $A+E$ be diagonalisable with eigendecompositions $A = X\Lambda X^{-1}$ and $A+E = \hat{X}\hat{\Lambda}\hat{X}^{-1}$, respectively. The eigenvalues are

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, \qquad \hat{\Lambda} = \begin{pmatrix} \hat{\lambda}_1 & & \\ & \ddots & \\ & & \hat{\lambda}_n \end{pmatrix}.$$

The Hoffman-Wielandt Theorem for diagonalisable matrices [6, Theorem 3.1] establishes a one-to-one pairing between exact and perturbed eigenvalues and bounds the sum of absolute errors in the Frobenius norm,

$$\sqrt{\sum_{i=1}^{n} |\lambda_i - \hat{\lambda}_{\tau(i)}|^2} \leq \kappa(\hat{X})\,\kappa(X)\ \|E\|_F \tag{5.1}$$

for some permutation $\tau$. Note that $\kappa(X)$ and $\kappa(\hat{X})$ are expressed in the two-norm, rather than in the Frobenius norm. This makes the bound tighter because the two-norm never exceeds the Frobenius norm.

To demonstrate that an absolute Hoffman-Wielandt-type bound implies a relative version, we need an absolute bound that is slightly stronger than (5.1). We replace $A$ by a product $AC$. The perturbed matrix is $AC+E$, where $C$ must have the same eigenvector matrix as $AC+E$. The bound (5.1) is the special case where $C = I$. The eigendecomposition of $C$ is

$$C = \hat{X}\Gamma\hat{X}^{-1}, \qquad \text{where} \qquad \Gamma = \begin{pmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_n \end{pmatrix}.$$

The eigendecompositions of $A$ and the perturbed matrix remain the same,

$$A = X\Lambda X^{-1}, \qquad AC + E = \hat{X}\hat{\Lambda}\hat{X}^{-1}.$$

The stronger Hoffman-Wielandt Theorem below bounds the sum of absolute errors in the products of the eigenvalues of $A$ and $C$.

THEOREM 5.1. *Let $A$, $C$ and $AC+E$ be diagonalisable. There exists a permutation $\tau$ so that*

$$\sqrt{\sum_{i=1}^{n} |\lambda_i \gamma_{\tau(i)} - \hat{\lambda}_{\tau(i)}|^2} \leq \kappa(\hat{X})\,\kappa(X)\ \|E\|_F.$$

*Proof.* The proof is similar to that of [6, Theorem 3.1].

In $AC - \hat{X}\hat{\Lambda}\hat{X}^{-1} = -E$ decompose $A$ and $C$,

$$X\Lambda X^{-1}\,\hat{X}\Gamma\hat{X}^{-1} - \hat{X}\hat{\Lambda}\hat{X}^{-1} = -E.$$

Multiply on the left by $X^{-1}$ and on the right by $\hat{X}$ and set $Z \equiv X^{-1}\hat{X}$,

$$\Lambda Z\Gamma - Z\hat{\Lambda} = -X^{-1}E\hat{X}. \tag{5.2}$$

The $(i,j)$th element of this equation has absolute value

$$|z_{ij}|\,|\lambda_i\gamma_j - \hat{\lambda}_j| = \left|(X^{-1}E\hat{X})_{ij}\right|.$$

The Frobenius norm is the sum of the squares of all these elements,

$$\sum_{i,j} |z_{ij}|^2\,|\lambda_i\gamma_j - \hat{\lambda}_j|^2 = \|X^{-1}E\hat{X}\|_F^2 \leq \|X^{-1}\|^2\,\|\hat{X}\|^2\,\|E\|_F^2.$$

By [6, Main Theorem], there exists a doubly stochastic matrix $S = (s_{ij})$ so that

$$\frac{s_{ij}}{\|Z^{-1}\|^2} \leq |z_{ij}|^2, \qquad 1 \leq i,j \leq n.$$

Hence

$$\begin{aligned} \sum_{i,j} s_{ij}\,|\lambda_i\gamma_j - \hat{\lambda}_j|^2 &\leq \|Z^{-1}\|^2 \sum_{i,j} |z_{ij}|^2\,|\lambda_i\gamma_j - \hat{\lambda}_j|^2 \\ &\leq \kappa(X)^2\,\kappa(\hat{X})^2\,\|E\|_F^2. \end{aligned}$$

Because $S$ is doubly stochastic, Birkhoff's theorem [3, §II.2] implies that there exists a permutation $\tau$ with

$$\sum_i |\lambda_i\gamma_{\tau(i)} - \hat{\lambda}_{\tau(i)}|^2 \leq \sum_{i,j} s_{ij}\,|\lambda_i\gamma_j - \hat{\lambda}_j|^2.$$

Therefore

$$\sum_i |\lambda_i\gamma_{\tau(i)} - \hat{\lambda}_{\tau(i)}|^2 \leq \kappa(X)^2\,\kappa(\hat{X})^2\,\|E\|_F^2.$$

□

The stronger absolute bound in Theorem 5.1 implies a relative bound, provided $A$ is non-singular. This relative bound is not new. It follows, for instance, from the multiplicative bound [12, Theorem 2.1'] with $D_1 = I$ and $D_2 = I + A^{-1}E$. However, the proof below demonstrates that the relative bound is no stronger than the absolute bound because it is implied by the absolute bound.

Corollary 5.2. *Let $A$ and $A+E$ be diagonalisable. If $A$ is also non-singular then there exists a permutation $\tau$ so that*

$$\sqrt{\sum_{i=1}^n \left(\frac{|\lambda_i - \hat{\lambda}_{\tau(i)}|}{|\lambda_i|}\right)^2} \leq \kappa(\hat{X})\,\kappa(X)\,\|A^{-1}E\|_F.$$

*Proof.* Since $A^{-1}(A+E) - A^{-1}E = I$ we can set

$$\bar{A} \equiv A^{-1}, \qquad C \equiv A + E, \qquad \bar{E} \equiv -A^{-1}E.$$

Then $\bar{A}$ is diagonalisable with eigenvector matrix $X$ and eigenvalues $\lambda_i^{-1}$; $C$ is diagonalisable with eigenvector matrix $\hat{X}$ and eigenvalues $\hat{\lambda}_i$; and $\bar{A}C + \bar{E} = \hat{X}I\hat{X}^{-1}$ is diagonalisable, where the eigenvalues are 1 and one can choose $\hat{X}$ as an eigenvector matrix. Applying Theorem 5.1 to $\bar{A}$, $C$ and $\bar{E}$ gives

$$\sum_{i=1}^n |\lambda_i^{-1}\hat{\lambda}_{\tau(i)} - 1|^2 \leq \kappa(\hat{X})^2\kappa(X)^2\|A^{-1}E\|_F^2.$$

□

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