IF A MATRIX HAS ONLY A SINGLE EIGENVALUE HOW SENSITIVE IS THIS EIGENVALUE? II

GRACE E. CHO* AND ILSE C.F. IPSEN†

January 22, 1998

Abstract. For matrices with a single eigenvalue the sensitivity of the eigenvalue to perturbations in the matrix is analyzed. Simple bounds are derived that separate the influence of the index on the eigenvalue sensitivity from the influence of the non-normality. The bounds suggest that the eigenvalue error grows proportional to the departure from normality when the index is large, while a small index weakens the influence of the non-normality on the eigenvalue error.

Numerical experiments with matrices that are diagonally similar to a Jordan matrix illustrate that the bounds accurately predict the behaviour of the error. It is also shown how to determine a 'refined' Schur decomposition of a Jordan matrix.

Key words. eigenvalue, defectiveness, departure from normality, Jordan decomposition, Schur decomposition, index

AMS subject classifications. 15A18, 15A42, 65F15, 65F35

1. Introduction.

We continue the work in [1], where we derived eigenvalue perturbation results for matrices that have a single eigenvalue. A matrix with a single eigenvalue λ has a Jordan decomposition $A = XJX^{-1}$, where all diagonal elements of J are equal to λ. In [1, Corollary 5.1] we used a refined Schur decomposition to bound the difference between λ and an eigenvalue of $A + E$, because the resulting bounds can be much tighter than those based on a Jordan decomposition. The bounds are expressed in terms of $\|E\|$, the departure of A from normality and the index of λ (the index is the size of a largest Jordan block and it is a measure for the defectiveness of the eigenvalue). Our goal here is to achieve a better understanding of these bounds. In particular, we want to separate the influence of the index on the eigenvalue sensitivity from the influence of the non-normality.

We distinguish between two types of non-normal matrices: weakly and highly non-normal. Based on a bound by Elsner [2], we showed in [1, §6] that an eigenvalue of a weakly non-normal matrix is insensitive to perturbations. It basically behaves like an eigenvalue of a normal matrix, and the index has no effect on the eigenvalue sensitivity. In the context of highly non-normal matrices, we simplify the bound [1, Corollary 5.1]. The simplified bounds suggest that the eigenvalue error grows proportional to the departure from normality when the index is large (where the definition of large is in reference to the departure from normality), while a small index weakens the influence of the non-normality on the eigenvalue error (§2).

For the numerical experiments we selected a class of matrices whose index and departure from normality are easy to control: matrices that are diagonally similar to a Jordan matrix. To determine their refined Schur decomposition we first derive a refined Schur decomposition of a Jordan matrix (§3). The numerical experiments illustrate that the bounds in §2 are tight for this particular class of matrices, and

*Department of Mathematics, North Carolina State University, P. O. Box 8205, Raleigh, NC 27695-8205, USA (ehcho@eos.ncsu.edu, http://www4.ncsu.edu/~ehcho/home.html).

†Department of Mathematics, North Carolina State University, P. O. Box 8205, Raleigh, NC 27695-8205, USA (ipsen@math.ncsu.edu, http://www4.ncsu.edu/~ipsen/info.html). The research of this author was supported in part by NSF grant CCR-9400921.

that they accurately estimate the influence of the index and of the departure from normality on the eigenvalue error (§4).

2. Interaction between Non-Normality and Index. We simplify the bound [1, Corollary 5.1] in order to separate the influence of the non-normality on the eigenvalue sensitivity from the influence of the index.

Our bounds are based on a Schur decomposition. A complex matrix A with a single eigenvalue λ has a Schur decomposition $A = Q(\lambda I + N)Q^*$, where Q is unitary and N is strictly upper triangular. Here I is the identity matrix, and the superscript * denotes the conjugate transpose. Although the non-normal part N is not unique, the departure from normality $\|N\|_F$ in the Frobenius norm is [5, §1.2]. Here we measure the departure from normality in the two-norm, $\|N\|$. Since

$$\|N\| \le \|N\|_F \le \sqrt{n}\,\|N\|$$

for a matrix of order n [3, (2.3.7)], our measure of non-normality is never more than $\sqrt{n}$ away from the unique quantity $\|N\|_F$, regardless of the choice of Schur decomposition.

We distinguish two types of non-normal matrices: weakly and highly non-normal. In [1, §6] we called a matrix *weakly non-normal* if $\|N\|$ is on the order of machine precision. We call a matrix highly non-normal if $\|N\| \gg 1$.

Let μ be an eigenvalue of $A + E$. In [1, §6] we used a bound by Elsner [2, Satz 3],

$$|\lambda - \mu| \le \|E\| + \|N\|, \tag{2.1}$$

to argue that the eigenvalue of a weakly non-normal matrix is insensitive to perturbations, and that the index has essentially no effect on the sensitivity.

However, as Elsner already observed [2, §3], this bound is not necessarily tight. This can happen in particular when the matrix is highly non-normal.

Example 2.1. *Let*

$$A = \begin{pmatrix} \lambda & \epsilon^{-1} \\ & \lambda \end{pmatrix}, \qquad A + E = \begin{pmatrix} \lambda & \epsilon^{-1} \\ \epsilon & \lambda \end{pmatrix},$$

where $0 < \epsilon < 1$. The departure from normality of A is $\|N\| = \epsilon^{-1}$, and the perturbation is $\|E\| = \epsilon$. The error in any eigenvalue μ of $A + E$ is $|\lambda - \mu| = 1$, whereas (2.1) gives the larger bound $|\lambda - \mu| \le \epsilon + \epsilon^{-1}$.

Since (2.1) can be too pessimistic for highly non-normal matrices, we derive a more suitable bound for this case. To this end we use a 'refined Schur decomposition' [4, §10], [6], [10],

$$A = Q(\lambda I + N)Q^*,$$

where Q is unitary and

$$\lambda I + N = \begin{pmatrix} \lambda I_1 & * & \dots & \dots & * \\ & \lambda I_2 & * & \dots & * \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & * \\ & & & & \lambda I_s \end{pmatrix} \tag{2.2}$$

is block upper triangular. Here s is the index of λ, i.e., the size of a largest Jordan block in a Jordan form of A. The identity matrices I_k are of order n_k, where n_k is the number of linearly independent generalized eigenvectors of A of grade $s-k+1$. In particular, n_s is the geometric multiplicity of λ. The I_k are placed along the diagonal in increasing order,

$$1 \leq n_1 \leq n_2 \leq \ldots \leq n_s.$$

The vector $(n_s \quad n_{s-1} \quad \ldots \quad n_1)$ is the *Weyr characteristic corresponding to* λ [11, pp 114]. The triangular matrix $\lambda I + N$ in (2.2) is, apart from a permutation, in *Weyr form* [11, Definition 3.3].

In particular, if A is nondefective (semisimple), then $s = 1$ and $A = \lambda I$ is a normal matrix. If A is maximally defective (nonderogatory), then $s = n$ and $n_k = 1$, $1 \leq k \leq n$. In this case the absolute values of the entries of N are uniquely determined, regardless of the choice of Schur decomposition [8, Theorem 3], [11, Remark, p 119].

The refined Schur decomposition can be used to improve a bound by Henrici [5, Theorem 4] for the special case of matrices with a single eigenvalue [1, Corollary 5.1],

$$\frac{|\lambda-\mu|^s}{\|N^{s-1}\| + |\lambda-\mu|\,\|N^{s-2}\| + \ldots + |\lambda-\mu|^{s-1}} \leq \|E\|. \tag{2.3}$$

Since we do not find this bound is easy to interpret, we simplify it. The simplification is motivated by the following example.

EXAMPLE 2.2. *Let*

$$C = \begin{pmatrix} \lambda & \eta & & \\ & \ddots & \ddots & \\ & & \ddots & \eta \\ & & & \lambda \end{pmatrix}, \qquad C + E = \begin{pmatrix} \lambda & \eta & & \\ & \ddots & \ddots & \\ & & \ddots & \eta \\ \epsilon & & & \lambda \end{pmatrix}$$

be matrices of order n with $0 \leq \eta$ and $0 \leq \epsilon < 1$. The error in any eigenvalue μ of $C+E$ equals

$$|\lambda - \mu| = (\epsilon\eta^{n-1})^{1/n}.$$

Since $\|N\| = \eta$ and $\|E\| = \epsilon$ this error can be written as

$$|\lambda - \mu| = (\|E\|\,\|N\|^{n-1})^{1/n}. \tag{2.4}$$

We simplify the right-hand side in (2.3) by bounding it by a small multiple of (2.4). To this end we assume $|\lambda-\mu| \leq \|N\|$. Because Elsner's bound (2.1) guarantees $|\lambda - \mu| \leq \|E\| + \|N\|$, this is a reasonable assumption, provided $\|E\|$ is sufficiently small and the matrix is at least moderately non-normal, $\|N\| \gg \|E\|$. The simplified bound below follows from (2.3) as well as from [5, Theorem 4].

THEOREM 2.1. *Let $A = Q(\lambda I + N)Q^*$ be a refined Schur decomposition (2.2). If $\|N\| > 0$ and $\|N\| \geq |\lambda - \mu|$ then*

$$|\lambda - \mu| \leq \; s^{1/s}\,(\|E\|\,\|N\|^{s-1})^{1/s}.$$

Proof. Set $\eta \equiv \|N\|$ and $\epsilon \equiv \|E\|$. The bound (2.3) implies

$$|\lambda-\mu|^s \leq \epsilon \sum_{k=0}^{s-1} |\lambda-\mu|^k \eta^{s-1-k} = \epsilon\eta^{s-1} \sum_{k=0}^{s-1} \left(\frac{|\lambda-\mu|}{\eta}\right)^k \leq s\epsilon\eta^{s-1}.$$

□

Applying Theorem 2.1 to the matrix A in Example 2.1 gives

$$|\lambda - \mu| \leq \sqrt{2}.$$

The bound has the correct order of magnitude and differs from the true error only by a factor of $\sqrt{2}$.

Applying Theorem 2.1 to the matrix C in Example 2.2 with $\epsilon < \eta$ gives

$$|\lambda - \mu| \leq n^{1/n} (\epsilon\, \eta^{n-1})^{1/n}.$$

The bound differs from the true error only by a factor of $n^{1/n}$. Since $s^{1/s}$ approaches 1 as s increases, the bound from Theorem 2.1 becomes tighter as n grows. Contrast this with the extended Bauer-Fike Theorem [7, Theorem 2], [12, Theorem IV.1.12] applied to C. Let $C = XJX^{-1}$ be a Jordan decomposition. When $\eta > 1$ the Jordan condition number is

$$\kappa(X) \equiv \|X\|\, \|X^{-1}\| = \eta^{n-1},$$

and the extended Bauer-Fike theorem gives

$$|\lambda - \mu| \leq (\epsilon\, \eta^{n-1})^{1/n} \left(\sum_{k=0}^{n-1} |\lambda - \mu|^k \right)^{1/n}.$$

The Bauer-Fike theorem is tighter than Theorem 2.1 only for those matrices with $1 < \eta < 1/\sqrt[n-1]{\epsilon}$ (so that $|\lambda - \mu| < 1$). For all other moderately to highly non-normal matrices, i.e. $\eta > 1/\sqrt[n-1]{\epsilon}$, Theorem 2.1 is tighter than the Bauer-Fike theorem.

When the departure from normality is at least moderate, the bound in Theorem 2.1 can be simplified further.

COROLLARY 2.2. *If, in addition to the conditions of Theorem 2.1,* $\|N\| \geq 1$ *then*

$$|\lambda - \mu| \leq s^{1/s}\, \|E\|^{1/s}\, \|N\|.$$

The bound in Corollary 2.2 grows in proportion to the departure from normality. It is as tight as the one in Theorem 2.1 if s is large enough with reference to $\|N\|$, that is, if $\|N\|^{(s-1)/s} \approx \|N\|$. Below we specify more carefully what this means by requesting that $\|N\|^{(s-1)/s} \geq .1\, \|N\|$. This ensures that the bound in Corollary 2.2 differs from the one in Theorem 2.1 by a factor of at most 10.

REMARK 2.1. *In addition to the conditions of Theorem 2.1 let also* $\|N\| \geq 1$.

If $\|N\|^{1/s} \leq 10$ *then*

$$|\lambda - \mu| \leq s^{1/s}\, \|E\|^{1/s}\, \|N\|,$$

where

$$\|E\|^{1/s}\, \|N\| \leq 10\, (\|E\|\, \|N\|^{s-1})^{1/s}.$$

Therefore if the index of a highly non-normal matrix is large in reference to its departure from normality, i.e. $\|N\|^{1/s} \leq 10$, then the error bound grows proportional with the departure from normality. If the index is small, i.e. $\|N\|^{1/s} \gg 10$, then the index dampens the impact of the non-normality in the bound.

To illustrate that the bound in Corollary 2.1 can be tight we perform numerical experiments on a simple class of matrices, namely those matrices that are diagonally similar to a Jordan matrix. In order to do so, we first need to determine a refined Schur decomposition of a Jordan matrix.

3. Schur decomposition of a Jordan Matrix. We show that a refined Schur decomposition of a Jordan matrix is merely a symmetric permutation, where eigenvalues associated with eigenvectors of the same grade are placed next to each other.

Let J be a Jordan matrix of size n with a single eigenvalue λ. We want to find a refined Schur decomposition $J = QTQ^*$ where $T = \lambda I + N$ is as in (2.2). Since one can choose an orthonormal basis for a generalized left eigenspace of J from columns of the identity matrix, we construct Q as a permutation matrix that permutes J to a matrix T with the structure (2.2).

We describe Q and T with the notation from [9]. Let J be a block diagonal matrix

$$J = \mathrm{diag}\,(\, J_1^1 \quad \dots \quad J_1^{r_1} \quad | \quad \dots \qquad \dots \quad | \quad J_t^1 \quad \dots \quad J_t^{r_t}\,), \tag{3.1}$$

where

$$J_i^j = \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}, \qquad 1 \le j \le r_i, \quad 1 \le i \le t,$$

is a Jordan block. There are r_i Jordan blocks of order s_i, and t different orders s_i, $1 \le i \le t$. The Jordan blocks are placed along the diagonal in decreasing order,

$$s_1 > s_2 > \ldots > s_t \ge 1.$$

The index s of λ is the size of a largest Jordan block, $s = s_1$.

Partition the Schur vectors Q conformally with (2.2),

$$Q = (\, Q_1 \quad \dots \quad Q_s\,)\,. \tag{3.2}$$

The columns of the $n \times n_k$ matrix Q_k represent an orthonormal basis for the generalized left eigenspace of A of grade $s - k + 1$. The following notation helps to count the columns in Q_i. Define

$$f_i \equiv r_1 + \ldots + r_i, \qquad 1 \le i \le t.$$

The geometric multiplicity g of λ, i.e., the number of linearly independent eigenvectors of λ, is the total number of Jordan blocks,

$$g \equiv f_t = r_1 + \ldots + r_t.$$

Each Jordan block has one eigenvector. A Jordan block of size s_t (the Jordan block of smallest size) cannot have a generalized eigenvector of grade larger than s_t. Thus f_t is also the number of generalized eigenvectors of grade k for $1 \le k \le s_t$. In general, Jordan blocks of size s_i or larger have generalized eigenvectors of grade s_i, while Jordan blocks of size less than s_i do not. Thus, f_i is the number of generalized eigenvectors of grade s_i. Since a Jordan block of size s_i cannot have a generalized eigenvector of grade larger than s_i, f_i is also the number of eigenvectors of grade k where $s_{i+1} < k \le s_i$ ($s_{t+1} \equiv 0$). Since the columns of Q_{s-k+1} are generalized left eigenvectors of grade k, Q_{s-k+1} has f_i columns where i is such that $s_{i+1} < k \le s_i$.

Denote by $m(l)$ the end position of the l-*th* Jordan block from the top. Decomposing

$$l = f_{i-1} + j, \qquad 1 \le j \le r_i, \quad 1 \le i \le t,$$

allows us to write

$$m(l) = m(f_{i-1} + j) = s_1 r_1 + \ldots + s_{i-1} r_{i-1} + j s_i,$$

where $s_0 = r_0 = f_0 = 0$. The canonical vector $e_{m(l)}$ is a left eigenvector of J, and $e_{m(l)-1}$ is a generalized left eigenvector of grade 2 of J. In general, $e_{m(l)-k+1}$ is a generalized left eigenvector of grade k of J.

Before stating the theorem formally, we present an example that illustrates how to obtain a refined Schur decomposition of a Jordan matrix.

EXAMPLE 3.1. *Let J have two Jordan blocks of size 4 and one of size 2. To distinguish the copies of λ, we write*

$$J = \left(\begin{array}{cccc|cccc|cc}
\lambda_1 & 1 & & & & & & & & \\
 & \lambda_2 & 1 & & & & & & & \\
 & & \lambda_3 & 1 & & & & & & \\
 & & & \lambda_4 & & & & & & \\
\hline
 & & & & \lambda_5 & 1 & & & & \\
 & & & & & \lambda_6 & 1 & & & \\
 & & & & & & \lambda_7 & 1 & & \\
 & & & & & & & \lambda_8 & & \\
\hline
 & & & & & & & & \lambda_9 & 1 \\
 & & & & & & & & & \lambda_{10}
\end{array}\right).$$

The geometric multiplicity is $g = 3$ and the index is $s = 4$. With the notation above

$$\begin{aligned} s_1 &= 4, \quad r_1 = 2, \quad f_1 = 2, \\ s_2 &= 2, \quad r_2 = 1, \quad f_2 = 3, \end{aligned}$$

and

$$m(1) = 4, \quad m(2) = 8, \quad m(3) = 10.$$

To determine a refined Schur decomposition, construct a permutation matrix Q as follows. The last three rows of Q^* are left eigenvectors of J, and the rows above these are generalized left eigenvectors of grade 2. Thus the grade of the generalized left eigenvectors in Q^* increases from bottom to top.

To count the columns in Q_{s-k+1}, start with $k = 1$, where $Q_4 = Q_{s-k+1}$ has $f_2 = 3$ columns. Next is $k = 2$, where $Q_3 = Q_{s-k+1}$ also has $f_2 = 3$ columns because $k \leq s_2$. When $k = 3$ then $Q_2 = Q_{s-k+1}$ has $f_1 = 2$ columns because $s_2 < k \leq s_1$. At last $k = 4 = s_1$, where $Q_1 = Q_{s-k+1}$ has $f_1 = 2$ columns. The resulting permutation matrix is

$$Q = \left(\begin{array}{cc|cc|ccc|ccc} e_1 & e_5 & e_2 & e_6 & e_3 & e_7 & e_9 & e_4 & e_8 & e_{10} \end{array}\right).$$

We determine $Q^* J Q$ by first permuting the columns,

$$JQ = \left(\begin{array}{cc|cc|ccc|ccc}
\lambda_1 & & 1 & & & & & & & \\
 & & \lambda_2 & & 1 & & & & & \\
 & & & & \lambda_3 & & & 1 & & \\
 & & & & & & & \lambda_4 & & \\
 & \lambda_5 & & 1 & & & & & & \\
 & & & \lambda_6 & & 1 & & & & \\
 & & & & & \lambda_7 & & & 1 & \\
 & & & & & & & & \lambda_8 & \\
 & & & & & & \lambda_9 & & & 1 \\
 & & & & & & & & & \lambda_{10}
\end{array}\right).$$

and then the rows,

$$
Q^*JQ = \left(\begin{array}{cc|cc|ccc|ccc}
\lambda_1 & & 1 & & & & & & & \\
& \lambda_5 & & 1 & & & & & & \\
\hline
& & \lambda_2 & & 1 & & & & & \\
& & & \lambda_6 & & 1 & & & & \\
\hline
& & & & \lambda_3 & & & 1 & & \\
& & & & & \lambda_7 & & & 1 & \\
& & & & & & \lambda_9 & & & 1 \\
\hline
& & & & & & & \lambda_4 & & \\
& & & & & & & & \lambda_8 & \\
& & & & & & & & & \lambda_{10}
\end{array}\right).
$$

Now we are ready to determine a refined Schur decomposition (2.2) of a general Jordan matrix. Define

$$\widetilde{I}_{s-k+1} \equiv \begin{pmatrix} I_{s-k+1} & O \end{pmatrix}, \qquad s_{i+1} < k \leq s_i,$$

where I_{s-k+1} is the identity matrix of size f_i. When $k = s_{i+1} + 1$ the rectangular zero matrix O has $f_{i+1} - f_i$ columns, otherwise it is empty.

THEOREM 3.1. *The Jordan matrix J in (3.1) has a refined Schur decomposition $J = QTQ^*$ with the following properties.*

The Schur vectors $Q = \begin{pmatrix} Q_1 & \dots & Q_s \end{pmatrix}$ satisfy

$$Q_{s-k+1} = \begin{pmatrix} e_{m(1)-k+1} & e_{m(2)-k+1} & \dots & e_{m(f_i)-k+1} \end{pmatrix}, \tag{3.3}$$

where $s_{i+1} < k \leq s_i$ and $1 \leq k \leq s$.

The upper triangular matrix T is of the form

$$T = \begin{pmatrix} \lambda I_1 & \widetilde{I}_1 & & & \\ & \lambda I_2 & \widetilde{I}_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \widetilde{I}_{s-1} \\ & & & & \lambda I_s \end{pmatrix}. \tag{3.4}$$

Proof. We show that QTQ^*, where Q is as in (3.3) and T as in (3.4), is equal to J. To this end multiply

$$
\begin{aligned}
QTQ^* &= \begin{pmatrix} Q_1 & Q_2 & \dots & Q_s \end{pmatrix} \begin{pmatrix} \lambda I_1 & \widetilde{I}_1 & & & \\ & \lambda I_2 & \widetilde{I}_2 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \widetilde{I}_{s-1} \\ & & & & \lambda I_s \end{pmatrix} \begin{pmatrix} Q_1^* \\ Q_2^* \\ \vdots \\ \vdots \\ Q_s^* \end{pmatrix} \\
&= \begin{pmatrix} \lambda Q_1 & \widetilde{Q}_1 + \lambda Q_2 & \dots & \widetilde{Q}_{s-1} + \lambda Q_s \end{pmatrix} \begin{pmatrix} Q_1^* \\ Q_2^* \\ \vdots \\ \vdots \\ Q_s^* \end{pmatrix} \\
&= \lambda Q_1 Q_1^* + (\widetilde{Q}_1 + \lambda Q_2) Q_2^* + \dots + (\widetilde{Q}_{s-1} + \lambda Q_s) Q_s^*,
\end{aligned}
$$

where $\widetilde{Q}_k \equiv Q_k \widetilde{I}_k$. Since Q is unitary,

$$\begin{aligned} QTQ^* &= \lambda(Q_1Q_1^* + \ldots Q_sQ_s^*) + (\widetilde{Q}_1Q_2^* + \ldots \widetilde{Q}_{s-1}Q_s^*) \\ &= \lambda I + (\widetilde{Q}_1Q_2^* + \ldots \widetilde{Q}_{s-1}Q_s^*). \end{aligned}$$

The diagonal elements of Q^*TQ are all λ because QTQ^* is a symmetric permutation of T. Hence it suffices to show that the symmetric permutation places the ones in the same position as in J.

To evaluate the products $\widetilde{Q}_{s-k}Q_{s-k+1}^*$, identify the columns of

$$\begin{aligned} Q_{s-k+1}^* &= \begin{pmatrix} e_{m(1)-k+1}^* \\ e_{m(2)-k+1}^* \\ \vdots \\ e_{m(f_i)-k+1}^* \end{pmatrix} \\ &= (\,0 \;\; \ldots \;\; 0 \;\; e_1 \;\; 0 \;\; \ldots \;\; 0 \;\; e_2 \;\; 0 \;\; \ldots \;\; 0 \;\; e_{f_i} \;\; 0 \;\; \ldots \;\; 0\,), \end{aligned} \tag{3.5}$$

where e_i is in position $m(i) - k + 1$, and 0 represents a zero column. Hence

$$\begin{aligned} &\widetilde{Q}_{s-k}Q_{s-k+1}^* \\ &= (\,e_{m(1)-k} \;\; e_{m(2)-k} \;\; \cdots \;\; e_{m(f_i)-k} \;\; 0 \;\; \ldots \;\; 0\,) \\ &\qquad \cdot (\,0 \;\; \ldots \;\; 0 \;\; e_1 \;\; 0 \;\; \ldots \;\; 0 \;\; e_2 \;\; 0 \;\; \ldots \;\; 0 \;\; e_{f_i} \;\; 0 \;\; \ldots \;\; 0\,) \\ &= (\,0 \;\; \ldots \;\; 0 \;\; e_{m(1)-k} \;\; 0 \;\; \ldots \;\; 0 \;\; e_{m(2)-k} \;\; 0 \;\; \ldots \;\; 0 \;\; e_{m(f_i)-k} \;\; 0 \;\; \ldots \;\; 0\,) \end{aligned}$$

where $e_{m(j)-k}$ is in position $m(j) - k + 1$. This means, all ones in the matrix $\widetilde{Q}_{s-k}Q_{s-k+1}^*$ are on the superdiagonal. Furthermore, the positions of the ones are distinct. Thus $\widetilde{Q}_1Q_2^* + \ldots \widetilde{Q}_{s-1}Q_s^*$ has

$$n_1 + \ldots + n_{s-1} = n - n_s = n - g$$

ones on the superdiagonal. Finally, the zeros on the superdiagonal are in the correct positions, that is, they are in positions $(m(j), m(j+1))$, $1 \leq j \leq g-1$, because according to (3.5), column e_j is in position $(m(j) - k + 1)$ so that column $m(j) - k$ of $\widetilde{Q}_{s-k}Q_{s-k+1}^*$ is zero. Therefore $QTQ^* = J$. □

The above Schur decomposition leads directly to a refined Schur decomposition of a matrix C that is diagonally similar to a Jordan matrix,

$$C = \lambda I + \eta Z, \qquad \text{where} \qquad Z \equiv J - \lambda I$$

is the strictly upper triangular part of a Jordan matrix. Such a matrix appeared already in Example 2.2. We selected this particular class of matrices because index and departure from normality are easy to control.

Let $C = XJX^{-1}$ be a Jordan decomposition, where X is a diagonal matrix

$$X = \operatorname{diag}(\,X_1 \quad \ldots \quad X_g\,),$$

and

$$X_i = \begin{pmatrix} 1 & & & \\ & \eta & & \\ & & \ddots & \\ & & & \eta^{s_i-1} \end{pmatrix}.$$

As defined in §3, s_i is the order of the ith Jordan block. Let $J = Q(\lambda I + N)Q^*$ be a refined Schur decomposition as in Theorem 3.1, then

$$C = Q(\lambda I + \eta N)Q^*$$

is a refined Schur decomposition of C where $\lambda I + \eta N$ is of the form (2.2).

4. Numerical Experiments. We present numerical experiments to illustrate that the bounds in §2 can be tight and that they accurately predict the influence of index and non-normality on the eigenvalue sensitivity.

The test matrices are of the form $C = \lambda I + \eta Z$ and are diagonally similar to a Jordan matrix, as described at the end of the previous section. They have order $n = 32$ or $n = 100$, a single eigenvalue $\lambda = 0$, Schur vectors $Q = I$, and Jordan blocks of various sizes. The zero eigenvalue presents no restriction because we measure only absolute errors $|\lambda - \mu|$. Without loss of generality one can assume $Q = I$ because the bounds in §2 are invariant under multiplication with unitary matrices. The perturbation matrices E are random matrices whose entries come from a uniform distribution on the interval $(0.0, 1.0)$.

All computations are performed in MATLAB (version 4.2c.1) in double precision with IEEE arithmetic on an Intel Pentium 120 MHz processor. The eigenvalues μ of the perturbed matrix $C + E$ are computed with the MATLAB function **eig**. The computed eigenvalues should have a backward error of about $\epsilon_{mach} \equiv 2.2204e - 16$. This backward error becomes irrelevant if we make our perturbations much larger and normalize them so that

$$\epsilon \equiv \|E\| = 10^{-8} \approx \sqrt{\epsilon_{mach}}.$$

The 'exact' eigenvalue error is computed as follows. Each C is perturbed by 10 different random matrices E. For each E, we compute the maximal eigenvalue error $\max_\mu |\lambda - \mu|$, where the maximum ranges over all eigenvalues μ of $C + E$. The 'maximal eigenvalue error' in the Figures is the average of the maximal errors over all perturbations E. The reason why we compute an average rather than a maximum is this. Since the maximum of maximal eigenvalue errors could be much larger than the average, it would be closer to our bounds and therefore make our bounds looks better. To avoid this, we chose to compare the bounds to an average error. Fortunately, in the case of the test matrices, average and maximal errors are of same order of magnitude so that they cannot be distinguished on the logarithmic scale.

The accuracy of the bounds is evaluated in two ways: keep the non-normality η fixed and view the eigenvalue error $|\lambda - \mu|$ as a function of the index s; and keep the index s fixed and view the error as a function of non-normality η.

Fig.4.1 plots the exact eigenvalue error and the bounds in §2 as a function of index for matrices C of order $n = 32$, with five different values for the departure from normality, $\eta = 10^{-8},\ 10^{-4},\ 1,\ 10^4,\ 10^8$. The vertical axis represents the average of the maximal eigenvalue errors on a logarithmic scale, and the horizontal axis represents the index. There are three different types of curves. A solid curve represents the exact eigenvalue error, a dashed curve represents the bound in Theorem 2.1, and a dotted curve represents the bound in Corollary 2.2.

For $\eta = 10^{-8}$ the dashed curve is below the exact eigenvalue error. Therefore the condition $|\lambda - \mu| \leq \eta$ in Theorem 2.1 does not hold. The exact eigenvalue error does not differ much from ϵ, i.e., the eigenvalue is insensitive, and the index does not affect the eigenvalue error. Elsner's bound (2.1) explains this: $|\lambda - \mu| \leq 2\epsilon$. For

$\eta = 10^{-8}$ the dotted curve coincides with the dashed curve. For $\eta > 10^{-8}$ the solid curve clearly stays below η, which means that the eigenvalue error is less than $\|N\|$ and the assumptions of Theorem 2.1 are justified.

All of curves exhibit the same qualitative behaviour. They increase steeply for index values from 1 to about 7, and from then on they essentially level off. Once the curves have levelled off, dashed and dotted curves are almost indistinguishable. During the steep increase, the eigenvalue errors are considerably smaller than the departure from normality. The discussion following Corollary 2.2 explains these observations. When $\eta \gg 1$ and the index s is small, then $\eta^{(s-1)/s} \ll \eta$ so that the index dampens the impact of the non-normality on the sensitivity of the eigenvalue. When $\eta \gg 1$ and the index is large, then $\eta^{(s-1)/s} \approx \eta$ so that eigenvalue error is proportional to the departure from normality. The transition from steep increase to levelling off is predicted by Remark 2.1, it should commence as soon as s is large enough so that $\|N\|^{1/s} \leq 10$. For example, if $\eta = 10^8$ and $s \geq 8$, then Remark 2.1 implies that $\eta^{(s-1)/s}$ differs from η by a factor of at most 10. This means, once the index has increased beyond a value of 8, the eigenvalue error starts to level off and the index has little influence on the error.

The dashed curves differ from the solid curves by a factor of at most 2.1654. Thus Theorem 2.1 is almost tight for the test matrices.

Fig.4.3 shows a similar picture for matrices C of order $n = 100$ with the same values for departure from normality as in Fig.4.1.

Fig.4.2 plots the 'exact' eigenvalue error as a function of non-normality for matrices C of order $n = 32$, with four different values for the index, $s = 8, 11, 29, 32$. The vertical axis represents the average of the maximal eigenvalue errors in units of 10^7, while the horizontal axis contains 10 different values of non-normality ranging from 10^7 to 10^8. The eigenvalue error increases linearly with the non-normality. This agrees with Corollary 2.2, and it illustrates that non-normality is a multiplicative factor in the eigenvalue error when the matrix is highly non-normal.

Fig.4.4 shows a similar picture for matrices C of order $n = 100$ and index values $s = 8, 23, 85, 100$.

Acknowledgment. We thank Volker Mehrmann for bringing reference [11] to our attention.

REFERENCES

[1] G. Cho and I. Ipsen, *If a matrix has only a single eigenvalue, how sensitive is this eigenvalue?*, Tech. Report CRSC-TR97-20, Center for Research in Scientific Computation, Department of Mathematics, North Carolina State University, 1997. Submitted *SIAM J. Matrix Anal. Appl.*

[2] L. Elsner, *Einschliessungssätze für Eigenwerte nichtnormaler Matrizen*, Funktionalanalysis, Approximationstheorie, Numerische Mathematik, 7 (1967), pp. 185–193.

[3] G. Golub and C. van Loan, *Matrix Computations*, The Johns Hopkins University Press, Baltimore, third ed., 1996.

[4] G. Golub and J. Wilkinson, *Ill-conditioned eigensystems and the computation of the Jordan canonical form*, SIAM Review, 18 (1976), pp. 578–619.

[5] P. Henrici, *Bounds for iterates, inverses, spectral variation and fields of values of non-normal matrices*, Numer. Math., 14 (1962), pp. 24–40.

[6] B. Kågström and A. Ruhe, *An algorithm for numerical compuatation of the Jordan normal form of a complex matrix*, ACM Trans. Math. Software, 6 (1980), pp. 398–419.

[7] R. Li, *On perturbation bounds for eigenvalues of a matrix.* Manuscript, Computing Center, Academia Sinica, Beijing, P.R.China, 1985.

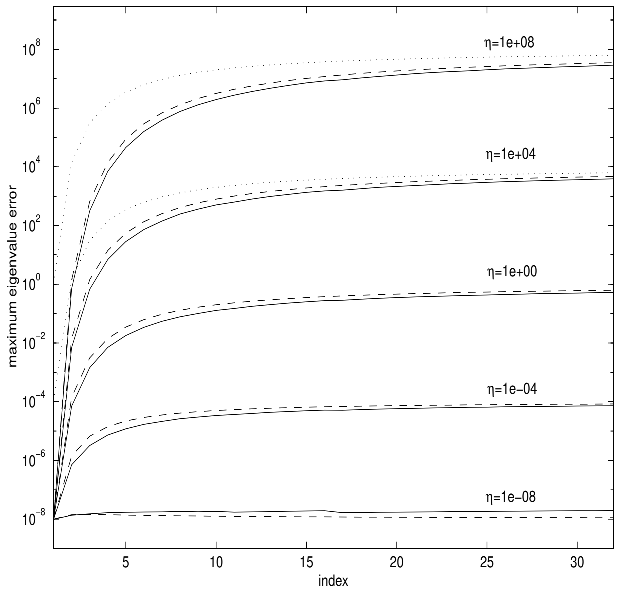

FIG. 4.1. *Eigenvalue error as a function of index(n=32)*

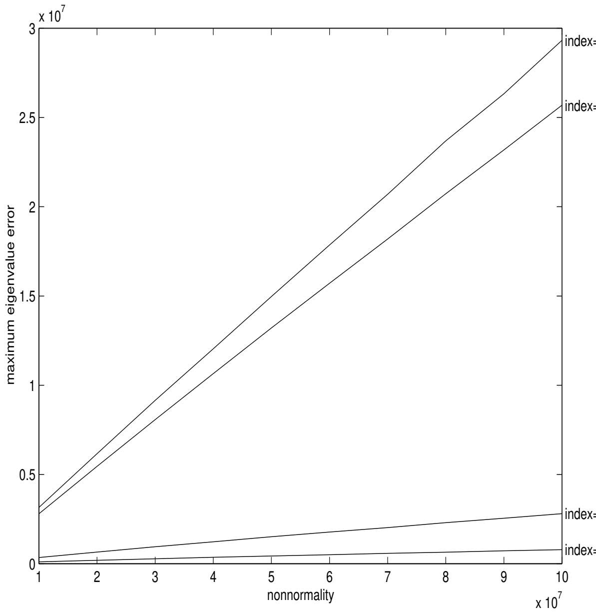

FIG. 4.2. *Eigenvalue error as a function of non-normality(n=32)*

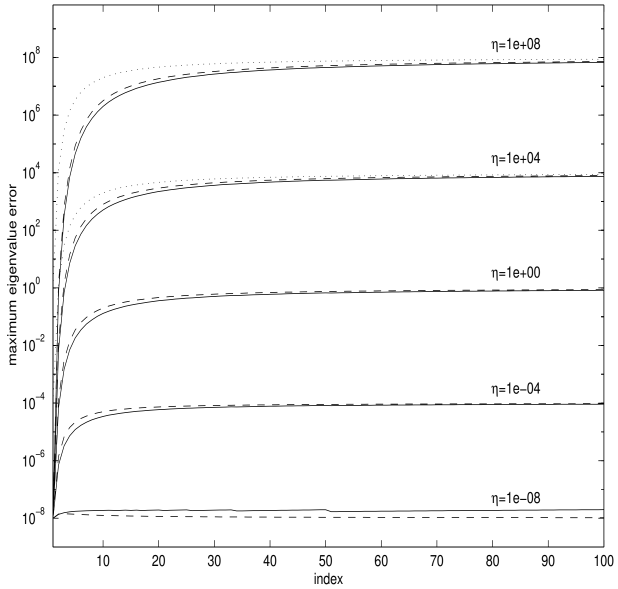

FIG. 4.3. *Eigenvalue error as a function of index(n=100)*

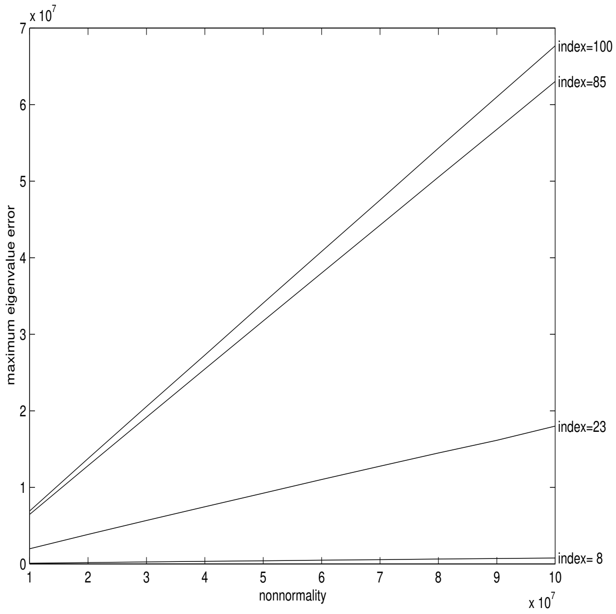

FIG. 4.4. *Eigenvalue error as a function of non-normality(n=100)*

[8] B. MITCHELL, *Unitary transformations*, Canad. J. Math., 6 (1954), pp. 69–72.
[9] J. MORO, J. BURKE, AND M. OVERTON, *On the Lidskii-Vishik-Lyusternik perturbation theory for eigenvalues of matrices with arbitrary Jordan structure*, SIAM J. Matrix Anal. Appl., 18 (1997), pp. 793–817.
[10] A. RUHE, *An algorithm for numerical determination of the structure of a general matrix*, BIT, 10 (1970), pp. 196–216.

[11] H. Shapiro, *A survey of canonical forms and invariants for unitary similarity*, Linear Algebra and its Applications, 147 (1991), pp. 101–167.
[12] G. Stewart and J. Sun, *Matrix Perturbation Theory*, Academic Press, San Diego, 1990.