The Angle Between Complementary Subspaces

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1. INTRODUCTION. Almost all linear algebra courses discuss angles between vectors. The angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathfrak{R}^n$ is defined as the number $0 \leq \theta \leq \pi/2$ that satisfies

$$\cos\theta = \mathbf{v}^T\mathbf{u}/\|\mathbf{v}\|_2\|\mathbf{u}\|_2.$$

Usually the discussion stops right there, and extensions to angles between subspaces of higher dimensions are, more or less tacitly, shoved under the rug. Perhaps this is because most instructors feel that such extensions are difficult to understand, or that further effort in this direction is not worthwhile. Indeed, this makes sense for angles between general subspaces because one would have to introduce concepts like *gap* or *distance* between subspaces [**7**, **12**], *principal* (*or canonical*) *angles* [**1**, **2**, **15**, **12**], the *CS decomposition* [**11**, **4**, **10**, **6**, **12**], and so on. These topics are better off in a more advanced course.

However, angles between *complementary* subspaces are easier to deal with. The purpose of our article is to draw attention to some simple, though not very well known, expressions for the angle between complementary subspaces which are easily derived from the fundamental theorem of linear algebra [**14**] and elementary facts about matrix norms and projectors.

Angles between complementary subspaces are not just academic. They arise, for instance, in the context of controller robustness [**9**, **16**]. Roughly speaking, the spaces associated with the controller and the plant (a system described by a set of differential equations) are complementary subspaces. The robustness of the controller is defined by the smallest perturbation that renders the system unstable, which means that the associated subspaces are no longer complementary. The system remains stable as long as perturbations are smaller than the distance between the complementary subspaces. One measure of distance is the sine of the angle between the spaces.

2. WHICH ANGLE? Before proving any theorems, we need to be precise about which angle we are talking about. As the dimension grows beyond $n > 2$, so does the wiggle room in $\mathfrak{R}^n$, and there are a host of different angles which can be defined between a pair of general subspaces. But since we wish to eventually concentrate on complementary spaces, the concept of the *minimal angle* is the most natural one to focus on.

Definition 2.1. *For nonzero subspaces $\mathcal{R}, \mathcal{N} \subseteq \Re^n$, the minimal angle between $\mathcal{R}$ and $\mathcal{N}$ is defined to be the number $0 \leq \theta \leq \pi/2$ that satisfies*

$$\cos\theta = \max_{\substack{\mathbf{u}\in\mathcal{R},\, \mathbf{v}\in\mathcal{N} \\ \|\mathbf{u}\|_2 = \|\mathbf{v}\|_2 = 1}} \mathbf{v}^T\mathbf{u}. \tag{2.1}$$

Notice that $\theta = 0$ if and only if $\mathcal{R} \cap \mathcal{N} \neq \mathbf{0}$, and $\theta = \pi/2$ if and only if $\mathcal{R} \perp \mathcal{N}$.

While (2.1) serves to define θ, it is not easy to use—especially if one wants to compute the value of θ for a given pair of subspaces. The trick in making θ more accessible is to first think in terms of projections, and then to shift the emphasis to $\sin\theta = (1 - \cos^2\theta)^{1/2}$.

The development also requires some elementary facts concerning the standard matrix 2-norm defined by

$$\|\mathbf{A}\|_2 = \max_{\|\mathbf{x}\|_2 = 1} \|\mathbf{A}\mathbf{x}\|_2 \quad \text{for } \mathbf{A} \in \Re^{n\times n} \quad \text{and} \quad \mathbf{x} \in \Re^{n\times 1}.$$

The following properties can be found (often as exercises) in standard texts.

$$\|\mathbf{A}^T\|_2 = \|\mathbf{A}\|_2 \tag{2.2}$$

$$\|\mathbf{XAY}\|_2 = \|\mathbf{A}\|_2 \text{ when } \mathbf{X} \text{ has orthonormal columns and } \mathbf{Y} \text{ has orthonormal rows} \tag{2.3}$$

$$\|\mathbf{A}\|_2 = \max_{\substack{\|\mathbf{x}\|_2 \leq 1 \\ \|\mathbf{y}\|_2 \leq 1}} \mathbf{y}^T\mathbf{A}\mathbf{x} \tag{2.4}$$

$$\|\mathbf{A}\|_2 = \frac{1}{\min\limits_{\|\mathbf{x}\|_2 = 1} \|\mathbf{A}^{-1}\mathbf{x}\|_2} \text{ when } \mathbf{A}^{-1} \text{ exists} \tag{2.5}$$

$$\left\| \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} \right\|_2 = \max\{\|\mathbf{A}\|_2, \|\mathbf{B}\|_2\}. \tag{2.6}$$

The first step in unraveling (2.1) is to express $\cos\theta$ in terms of the orthogonal projectors onto $\mathcal{R}$ and $\mathcal{N}$.

Theorem 2.1. *If $\mathbf{P}_{\mathcal{R}}$ and $\mathbf{P}_{\mathcal{N}}$ are the orthogonal projectors onto $\mathcal{R}$ and $\mathcal{N}$, respectively, then*

$$\cos\theta = \|\mathbf{P}_{\mathcal{N}}\mathbf{P}_{\mathcal{R}}\|_2 = \|\mathbf{P}_{\mathcal{R}}\mathbf{P}_{\mathcal{N}}\|_2. \tag{2.7}$$

Proof: For vectors $\mathbf{x}$ and $\mathbf{y}$ such that $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 = 1$, we have $\mathbf{P}_{\mathcal{R}}\mathbf{x} \in \mathcal{R}$ and $\mathbf{P}_{\mathcal{N}}\mathbf{y} \in \mathcal{N}$ where $\|\mathbf{P}_{\mathcal{R}}\mathbf{x}\|_2 \leq \|\mathbf{P}_{\mathcal{R}}\|_2\|\mathbf{x}\|_2 \leq 1$ and $\|\mathbf{P}_{\mathcal{N}}\mathbf{y}\|_2 \leq \|\mathbf{P}_{\mathcal{N}}\|_2\|\mathbf{y}\|_2 \leq 1$, so that (2.4) can be used to write

$$\cos\theta = \max_{\substack{\mathbf{u}\in\mathcal{R},\, \mathbf{v}\in\mathcal{N} \\ \|\mathbf{u}\|_2 = \|\mathbf{v}\|_2 = 1}} \mathbf{v}^T\mathbf{u} = \max_{\substack{\mathbf{u}\in\mathcal{R},\, \mathbf{v}\in\mathcal{N} \\ \|\mathbf{u}\|_2 \leq 1,\, \|\mathbf{v}\|_2 \leq 1}} \mathbf{v}^T\mathbf{u} = \max_{\substack{\|\mathbf{x}\|_2 \leq 1 \\ \|\mathbf{y}\|_2 \leq 1}} \mathbf{y}^T\mathbf{P}_{\mathcal{N}}\mathbf{P}_{\mathcal{R}}\mathbf{x} = \|\mathbf{P}_{\mathcal{N}}\mathbf{P}_{\mathcal{R}}\|_2.$$

The fact that $\|\mathbf{P}_{\mathcal{N}}\mathbf{P}_{\mathcal{R}}\|_2 = \|\mathbf{P}_{\mathcal{R}}\mathbf{P}_{\mathcal{N}}\|_2$ is a consequence of the symmetry of orthogonal projectors together with (2.2). ∎

Theorem 2.1 does not depend on $\mathcal{R}$ and $\mathcal{N}$ being complementary subspaces—it is a statement about the minimal angle between any two subspaces of $\Re^n$. But in the special case when $\mathcal{R}$ and $\mathcal{N}$ are complementary, there is a more natural projector which gives rise to a formula which is simpler than (2.7).

3. ENTER THE OBLIQUE PROJECTOR.

Definition 3.1. *Subspaces* $\mathcal{R}, \mathcal{N} \subseteq \Re^n$ *are said to be complementary whenever* $\mathcal{R} + \mathcal{N} = \Re^n$ *and* $\mathcal{R} \cap \mathcal{N} = \mathbf{0}$, *and this is denoted by writing* $\mathcal{R} \oplus \mathcal{N} = \Re^n$. *The associated oblique projector is the unique idempotent matrix* $\mathbf{P}$ *whose range is* $\mathcal{R}$ *and whose nullspace is* $\mathcal{N}$. *As an operator,* $\mathbf{P}$ *projects vectors in* $\Re^n$ *onto* $\mathcal{R}$ *along (or parallel to)* $\mathcal{N}$, *and thus it acts as the identity on* $\mathcal{R}$ *and the zero operator on* $\mathcal{N}$.

The goal is to simplify (2.7) in the case of complementary spaces by somehow using the more natural oblique projector $\mathbf{P}$ instead of the two orthogonal projectors $\mathbf{P}_{\mathcal{R}}$ and $\mathbf{P}_{\mathcal{N}}$. But to realize a simplification, we must shift the emphasis to $\sin\theta$ rather than $\cos\theta$.

Theorem 3.1. *Suppose that* $\mathcal{R}, \mathcal{N} \subset \Re^n$ *are nonzero complementary spaces, and let* $\mathbf{P}$ *be the oblique projector onto* $\mathcal{R}$ *along* $\mathcal{N}$. *The minimal angle* θ *between* $\mathcal{R}$ *and* $\mathcal{N}$ *satisfies*

$$\sin\theta = \frac{1}{\|\mathbf{P}\|_2}. \tag{3.1}$$

Proof: Decompose $\mathbf{P}$ in terms of its four fundamental subspaces by choosing orthogonal matrices $\mathbf{U} = (\mathbf{U}_1 \,|\, \mathbf{U}_2)$ and $\mathbf{V} = (\mathbf{V}_1 \,|\, \mathbf{V}_2)$ in which the columns of $\mathbf{U}_1$ and $\mathbf{U}_2$ constitute orthonormal bases for $\mathcal{R}$ and $\mathcal{R}^\perp$, respectively, and $\mathbf{V}_1$ and $\mathbf{V}_2$ are orthonormal bases for $\mathcal{N}^\perp$ and $\mathcal{N}$, respectively, so that $\mathbf{U}_i^T\mathbf{U}_i = \mathbf{I}$ and $\mathbf{V}_i^T\mathbf{V}_i = \mathbf{I}$ for $i = 1, 2$, and

$$\mathbf{P}_{\mathcal{R}} = \mathbf{U}_1\mathbf{U}_1^T, \quad \mathbf{I} - \mathbf{P}_{\mathcal{R}} = \mathbf{U}_2\mathbf{U}_2^T, \quad \mathbf{P}_{\mathcal{N}} = \mathbf{V}_2\mathbf{V}_2^T, \quad \mathbf{I} - \mathbf{P}_{\mathcal{N}} = \mathbf{V}_1\mathbf{V}_1^T.$$

The matrices $\mathbf{U}$ and $\mathbf{V}$ decompose $\mathbf{P}$ in the sense that

$$\mathbf{U}^T\mathbf{P}\mathbf{V} = \begin{pmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad \text{or, equivalently, } \mathbf{P} = \mathbf{U}\begin{pmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{V}^T = \mathbf{U}_1\mathbf{C}\mathbf{V}_1^T \tag{3.2}$$

in which $\mathbf{C} = \mathbf{U}_1^T\mathbf{P}\mathbf{V}_1$ is nonsingular. (For instance, one can choose $\mathbf{U}$ and $\mathbf{V}$ so that this is the singular value decomposition of $\mathbf{P}$.) Notice that $\mathbf{P}^2 = \mathbf{P}$ implies $\mathbf{C} = \mathbf{C}\mathbf{V}_1^T\mathbf{U}_1\mathbf{C}$, which in turn insures $\mathbf{C}^{-1} = \mathbf{V}_1^T\mathbf{U}_1$. Consequently, (2.3) together with (2.5) implies that

$$\|\mathbf{P}\|_2 = \|\mathbf{C}\|_2 = \frac{1}{\min\limits_{\|\mathbf{x}\|_2=1} \|\mathbf{C}^{-1}\mathbf{x}\|_2} = \frac{1}{\min\limits_{\|\mathbf{x}\|_2=1} \|\mathbf{V}_1^T\mathbf{U}_1\mathbf{x}\|_2}.$$

Combining this with the result of Theorem 2.1 produces

$$\begin{aligned}
\sin^2\theta &= 1 - \cos^2\theta = 1 - \|\mathbf{P}_{\mathcal{N}}\mathbf{P}_{\mathcal{R}}\|_2^2 = 1 - \|\mathbf{V}_2\mathbf{V}_2^T\mathbf{U}_1\mathbf{U}_1^T\|_2^2 \\
&= 1 - \left\|(\mathbf{I} - \mathbf{V}_1\mathbf{V}_1^T)\mathbf{U}_1\right\|_2^2 = 1 - \max_{\|\mathbf{x}\|_2=1} \left\|(\mathbf{I} - \mathbf{V}_1\mathbf{V}_1^T)\mathbf{U}_1\mathbf{x}\right\|_2^2 \\
&= 1 - \max_{\|\mathbf{x}\|_2=1} \mathbf{x}^T\mathbf{U}_1^T(\mathbf{I} - \mathbf{V}_1\mathbf{V}_1^T)\mathbf{U}_1\mathbf{x} = 1 - \max_{\|\mathbf{x}\|_2=1} \left(1 - \|\mathbf{V}_1^T\mathbf{U}_1\mathbf{x}\|_2^2\right) \\
&= 1 - \left(1 - \min_{\|\mathbf{x}\|_2=1} \|\mathbf{V}_1^T\mathbf{U}_1\mathbf{x}\|_2^2\right) \\
&= \frac{1}{\|\mathbf{P}\|_2^2}.
\end{aligned}$$

■

The expression $\sin\theta = 1/\|\mathbf{P}\|_2$ is not only conceptually simple, but, as illustrated in Figure 1, there is also a particularly nice picture that accompanies it. The image of the unit sphere in $\Re^3$ under $\mathbf{P}$ is obtained by projecting all vectors on the sphere onto $\mathscr{R}$ along lines parallel to $\mathscr{N}$. The result is an ellipse in $\mathscr{R}$.

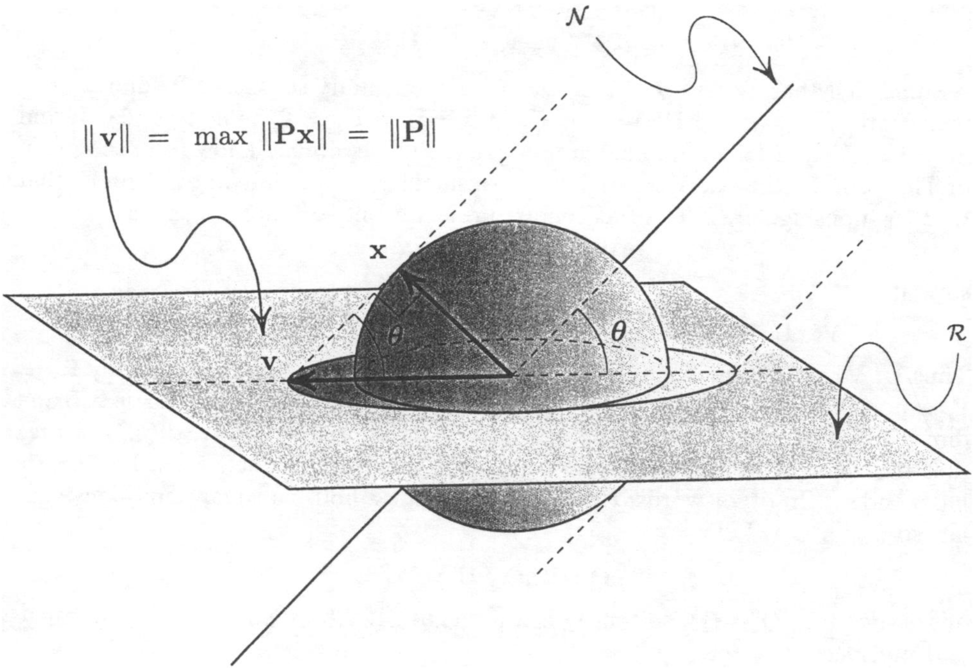

Figure 1

The norm of a longest vector $\mathbf{v}$ on this ellipse equals the norm of $\mathbf{P}$, i.e.

$$\|\mathbf{v}\|_2 = \max_{\|\mathbf{x}\|_2 = 1} \|\mathbf{Px}\|_2 = \|\mathbf{P}\|_2.$$

It is apparent from the right triangle in Figure 1 that

$$\sin\theta = \frac{\|\mathbf{x}\|_2}{\|\mathbf{v}\|_2} = \frac{1}{\|\mathbf{v}\|_2} = \frac{1}{\|\mathbf{P}\|_2}.$$

4. BACK TO ORTHOGONAL PROJECTORS. For subspaces $\mathscr{R}, \mathscr{N} \subseteq \Re^n$ such that $\dim \mathscr{R} = \dim \mathscr{N}$, the difference $\mathbf{P}_{\mathscr{R}} - \mathbf{P}_{\mathscr{N}}$ of the associated orthogonal projectors is of special interest because $\|\mathbf{P}_{\mathscr{R}} - \mathbf{P}_{\mathscr{N}}\|_2$ is a common measure of the distance or separation between $\mathscr{R}$ and $\mathscr{N}$. It is therefore natural to inquire about what can be said about the minimal angle between complementary spaces in terms of the difference $\mathbf{P}_{\mathscr{R}} - \mathbf{P}_{\mathscr{N}}$. The following theorem provides some answers.

Theorem 4.1. *For nonzero subspaces $\mathscr{R}, \mathscr{N} \subseteq \Re^n$, let $\mathbf{P}_{\mathscr{R}}$ and $\mathbf{P}_{\mathscr{N}}$ denote the orthogonal projectors onto $\mathscr{R}$ and $\mathscr{N}$, respectively, and let θ be the minimal angle between $\mathscr{R}$ and $\mathscr{N}$. The following two statements are true.*

- *$\mathscr{R}$ and $\mathscr{N}$ are complementary spaces if and only if $\mathbf{P}_{\mathscr{R}} - \mathbf{P}_{\mathscr{N}}$ is nonsingular.* (4.1)
- *If $\mathscr{R}$ and $\mathscr{N}$ are complementary spaces, then $\sin\theta = 1/\|(\mathbf{P}_{\mathscr{R}} - \mathbf{P}_{\mathscr{N}})^{-1}\|_2$.* (4.2)

Proof of (4.1): The orthogonal matrices $\mathbf{U}$ and $\mathbf{V}$ which were introduced in the proof of Theorem 3.1 to decompose $\mathbf{P}$ also decompose $\mathbf{P}_{\mathscr{R}} - \mathbf{P}_{\mathscr{N}}$ in the sense that

$$\mathbf{U}^T(\mathbf{P}_{\mathscr{R}} - \mathbf{P}_{\mathscr{N}})\mathbf{V} = \left(\frac{\mathbf{U}_1^T}{\mathbf{U}_2^T}\right)(\mathbf{U}_1\mathbf{U}_1^T - \mathbf{V}_2\mathbf{V}_2^T)(\mathbf{V}_1 \,|\, \mathbf{V}_2) = \begin{pmatrix} \mathbf{U}_1^T\mathbf{V}_1 & \mathbf{0} \\ \mathbf{0} & -\mathbf{U}_2^T\mathbf{V}_2 \end{pmatrix}. \tag{4.3}$$

Assume first that $\mathscr{R}$ and $\mathscr{N}$ are nonzero complementary subspaces. If $\dim \mathscr{R} = r$, then $\mathbf{U}_1^T\mathbf{V}_1$ is $r \times r$ and $\mathbf{U}_2^T\mathbf{V}_2$ is $n - r \times n - r$, so $\mathbf{P}_{\mathscr{R}} - \mathbf{P}_{\mathscr{N}}$ is nonsingular if and only if $\mathbf{U}_1^T\mathbf{V}_1$ and $\mathbf{U}_2^T\mathbf{V}_2$ are each nonsingular. But we already know from the proof of Theorem 3.1 that $\mathbf{U}_1^T\mathbf{V}_1 = (\mathbf{C}^{-1})^T$ is nonsingular, so we only need to prove that $\mathbf{U}_2^T\mathbf{V}_2$ is nonsingular. If $\mathbf{P}$ is the oblique projector onto $\mathscr{R}$ along $\mathscr{N}$, then

$$\mathbf{P}\mathbf{U}_1 = \mathbf{U}_1 \quad \text{and} \quad \mathbf{P}\mathbf{V}_2 = \mathbf{0},$$

so that

$$\mathbf{V}_2^T(\mathbf{I} - \mathbf{P})\mathbf{U}_2\mathbf{U}_2^T\mathbf{V}_2 = \mathbf{V}_2^T(\mathbf{I} - \mathbf{P})\big(\mathbf{I} - \mathbf{U}_1\mathbf{U}_1^T\big)\mathbf{V}_2 = \mathbf{V}_1^T\mathbf{V}_1 = \mathbf{I}.$$

Thus $\mathbf{U}_2^T\mathbf{V}_2$ is nonsingular with $(\mathbf{U}_2^T\mathbf{V}_2)^{-1} = \mathbf{V}_2^T(\mathbf{I} - \mathbf{P})\mathbf{U}_2$, and consequently $\mathbf{P}_{\mathscr{R}} - \mathbf{P}_{\mathscr{N}}$ is nonsingular. Conversely, if $\mathbf{P}_{\mathscr{R}} - \mathbf{P}_{\mathscr{N}}$ is nonsingular, and if $\dim \mathscr{R} = r > 0$ and $\dim \mathscr{N} = k > 0$, then $\mathbf{U}_1^T\mathbf{V}_1$ is $r \times n - k$ and $\mathbf{U}_2^T\mathbf{V}_2$ is $n - r \times k$, so (4.3) insures that the rows as well as the columns in each of these products must be linearly independent. In other words, $\mathbf{U}_1^T\mathbf{V}_1$ and $\mathbf{U}_2^T\mathbf{V}_2$ must both be square and nonsingular, so $k = n - r$. Let

$$\mathbf{Q} = \mathbf{U}_1\big(\mathbf{V}_1^T\mathbf{U}_1\big)^{-1}\mathbf{V}_1^T,$$

and notice that $\mathbf{Q} = \mathbf{Q}^2$, so that $\mathbf{Q}$ is a projector. If $R(*)$ and $N(*)$ denote range and nullspace, respectively, then

$$R(\mathbf{Q}) \subseteq R(\mathbf{U}_1) = R(\mathbf{Q}\mathbf{U}_1) \subseteq R(\mathbf{Q}) \Rightarrow R(\mathbf{Q}) = \mathscr{R},$$

and

$$\left\{\begin{matrix} N(\mathbf{Q}) \supseteq N\big(\mathbf{V}_1^T\big) = \mathscr{N} \\ \dim N(\mathbf{Q}) = n - \dim R(\mathbf{Q}) = n - r = k = \dim \mathscr{N} \end{matrix}\right\} \Rightarrow N(\mathbf{Q}) = \mathscr{N}.$$

In other words, $\mathbf{Q} = \mathbf{P}$ is the oblique projector onto $\mathscr{R}$ along $\mathscr{N}$. Therefore, since the range and nullspace of any projector are complementary spaces, it must be the case that $\mathscr{R} \oplus \mathscr{N} = \Re^n$. ■

Proof of (4.2): If $\mathscr{R}$ and $\mathscr{N}$ are complementary, then $\mathbf{P}_{\mathscr{R}} - \mathbf{P}_{\mathscr{N}}$ is nonsingular, and (4.3) together with (2.6) can be used to conclude that

$$\big\|(\mathbf{P}_{\mathscr{R}} - \mathbf{P}_{\mathscr{N}})^{-1}\big\|_2 = \max\Big\{\big\|\big(\mathbf{U}_1^T\mathbf{V}_1\big)^{-1}\big\|_2, \big\|\big(\mathbf{U}_2^T\mathbf{V}_2\big)^{-1}\big\|_2\Big\}. \tag{4.4}$$

But $\|(\mathbf{U}_1^T\mathbf{V}_1)^{-1}\|_2 = \|(\mathbf{U}_2^T\mathbf{V}_2)^{-1}\|_2$ because we can again use (2.5) to write

$$\begin{aligned} \frac{1}{\big\|\big(\mathbf{U}_1^T\mathbf{V}_1\big)^{-1}\big\|_2^2} &= \min_{\|\mathbf{x}\|_2 = 1} \|\mathbf{U}_1^T\mathbf{V}_1\mathbf{x}\|_2^2 = \min_{\|\mathbf{x}\|_2 = 1} \mathbf{x}^T\mathbf{V}_1^T\mathbf{U}_1\mathbf{U}_1^T\mathbf{V}_1\mathbf{x} \\ &= \min_{\|\mathbf{x}\|_2 = 1} \mathbf{x}^T\mathbf{V}_1^T\big(\mathbf{I} - \mathbf{U}_2\mathbf{U}_2^T\big)\mathbf{V}_1\mathbf{x} \\ &= \min_{\|\mathbf{x}\|_2 = 1} \big(1 - \mathbf{x}^T\mathbf{V}_1^T\mathbf{U}_2\mathbf{U}_2^T\mathbf{V}_1\mathbf{x}\big) \\ &= 1 - \max_{\|\mathbf{x}\|_2 = 1} \|\mathbf{U}_2^T\mathbf{V}_1\mathbf{x}\|_2^2 = 1 - \|\mathbf{U}_2^T\mathbf{V}_1\|_2^2, \end{aligned}$$

and a similar argument proves that

$$\frac{1}{\left\|\left(\mathbf{U}_2^T\mathbf{V}_2\right)^{-1}\right\|_2^2} = 1 - \|\mathbf{U}_2^T\mathbf{V}_1\|_2^2.$$

Therefore, the results of Theorem 3.1 insure that

$$\left\|(\mathbf{P}_{\mathscr{R}} - \mathbf{P}_{\mathscr{N}})^{-1}\right\|_2 = \left\|\left(\mathbf{U}_1^T\mathbf{V}_1\right)^{-1}\right\|_2 = \|\mathbf{C}^T\|_2 = \|\mathbf{C}\|_2 = \|\mathbf{P}\|_2 = \frac{1}{\sin\theta}. \qquad \blacksquare$$

Theorem 3.1 is not new—Gohberg and Kreĭn **[5]** attribute it to Ljance **[8]**—but it seems to have escaped the notice of many writers and teachers of linear algebra. We have not seen Theorem 4.1 in the literature.

5. CONSEQUENCES. Although the following facts about projectors are often proved by separate (and sometimes substantial) arguments, they turn out to be immediate consequences of Theorem 3.1 and Theorem 4.1.

Corollary 5.1. $\|\mathbf{P}\|_2 \geq 1$ *for every non-zero projector* $\mathbf{P}$, *and* $\|\mathbf{P}\|_2 = 1$ *if and only if* $\mathbf{P}$ *is an orthogonal projector.*

Corollary 5.2. $\|\mathbf{I} - \mathbf{P}\|_2 = \|\mathbf{P}\|_2$ *for all projectors* $\mathbf{P}$ *that are not zero and not equal to the identity.*

Corollary 5.3. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be vectors in* $\mathfrak{R}^n$ *with* $\mathbf{v}^T\mathbf{u} = 1$. *If* θ *is the minimal angle between* $\mathbf{u}$ *and* $\mathbf{v}^\perp$ *(the space orthogonal to* $\mathbf{v}$*), then*

$$\|\mathbf{I} - \mathbf{u}\mathbf{v}^T\|_2 = \|\mathbf{u}\mathbf{v}^T\|_2 = \frac{1}{\sin\theta} = \|\mathbf{u}\|_2\|\mathbf{v}\|_2.$$

Proof: The first equality follows from Corollary 5.2 and the second one from Theorem 3.1. The fact that $\|\mathbf{u}\mathbf{v}^T\|_2 = \|\mathbf{u}\|_2\|\mathbf{v}\|_2$ follows from properties of the two-norm because

$$\|\mathbf{u}\|_2\|\mathbf{v}\|_2 = \frac{\|\mathbf{u}\mathbf{v}^T\mathbf{v}\|_2}{\|\mathbf{v}\|_2} \leq \max_{\mathbf{x}\neq 0}\frac{\|\mathbf{u}\mathbf{v}^T\mathbf{x}\|_2}{\|\mathbf{x}\|_2} = \|\mathbf{u}\mathbf{v}^T\|_2 \leq \|\mathbf{u}\|_2\|\mathbf{v}\|_2. \qquad \blacksquare$$

Corollary 5.4. *If* θ *is the minimal angle between complementary spaces* $\mathscr{R},\mathscr{N} \subset \mathfrak{R}^n$, *and if* $\theta^\perp$ *is the minimal angle between* $\mathscr{R}^\perp$ *and* $\mathscr{N}^\perp$, *then* $\theta = \theta^\perp$.

Proof: This follows from Theorem 3.1 together with Corollary 5.2. The result is also a corollary of Theorem 4.1 because

$$\left\|(\mathbf{P}_{\mathscr{R}^\perp} - \mathbf{P}_{\mathscr{N}^\perp})^{-1}\right\|_2 = \left\|((\mathbf{I} - \mathbf{P}_{\mathscr{R}}) - (\mathbf{I} - \mathbf{P}_{\mathscr{N}}))^{-1}\right\|_2 = \left\|(\mathbf{P}_{\mathscr{R}} - \mathbf{P}_{\mathscr{N}})^{-1}\right\|_2.$$

Corollary 5.5. *For complementary spaces* $\mathscr{R},\mathscr{N} \subset \mathfrak{R}^n$, *let* $\mathbf{P}$ *be the oblique projector onto* $\mathscr{R}$ *along* $\mathscr{N}$, *and let* $\mathbf{Q}$ *denote the oblique projector onto* $\mathscr{R}^\perp$ *along* $\mathscr{N}^\perp$. *If* $\mathbf{P}_{\mathscr{R}}$ *and* $\mathbf{P}_{\mathscr{N}}$ *are the orthogonal projectors onto* $\mathscr{R}$ *and* $\mathscr{N}$, *respectively, and if* θ *is the minimal angle between* $\mathscr{R}$ *and* $\mathscr{N}$, *then each of the following statements is true.*

- $(\mathbf{P}_{\mathscr{R}} - \mathbf{P}_{\mathscr{N}})^{-1} = \mathbf{P} - \mathbf{Q}$
- $\sin\theta = \dfrac{1}{\|\mathbf{P} - \mathbf{Q}\|_2}$
- $\|\mathbf{P} - \mathbf{Q}\|_2 = \|\mathbf{P}\|_2$

Proof: The first equation can be derived from (4.3), or it can be verified by direct multiplication. The second and third equations follow from the first in conjunction with the results of Theorems 3.1 and 4.1. ∎

Corollary 5.6. *For complementary spaces $\mathscr{R},\mathscr{N} \subset \Re^n$, the oblique projector* $\mathbf{P}$ *onto* $\mathscr{R}$ *along* $\mathscr{N}$ *is given by the pseudoinverse of* $\mathbf{P}_{\mathscr{N}^\perp}\mathbf{P}_{\mathscr{R}}$ *where* $\mathbf{P}_{\mathscr{R}}$ *and* $\mathbf{P}_{\mathscr{N}^\perp}$ *are the orthogonal projectors onto* $\mathscr{R}$ *and* $\mathscr{N}^\perp$, *respectively. That is*

$$\mathbf{P} = (\mathbf{P}_{\mathscr{N}^\perp}\mathbf{P}_{\mathscr{R}})^\dagger.$$

Furthermore, if $\bar{\theta}$ *is the minimal angle between* $\mathscr{R}$ *and* $\mathscr{N}^\perp$, *then*

$$\cos\bar{\theta} = \|\mathbf{P}^\dagger\|_2.$$

Proof: To obtain the first equality, use (3.2) together with $\mathbf{C}^{-1} = \mathbf{V}_1^T\mathbf{U}_1$ to write

$$\mathbf{P}^\dagger = \mathbf{V}\begin{pmatrix} \mathbf{C}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{U}^T = \mathbf{V}_1\mathbf{C}^{-1}\mathbf{U}_1^T = \mathbf{V}_1\mathbf{V}_1^T\mathbf{U}_1\mathbf{U}_1^T = \mathbf{P}_{\mathscr{N}^\perp}\mathbf{P}_{\mathscr{R}}.$$

Now take the pseudoinverse of both sides (see **[3]** for details concerning pseudoinverses). The second equality is a consequence of the first in conjunction with Theorem 2.1. ∎

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Wc often hear that mathematics consists mainly of "proving theorems." Is a writer's job mainly that of "writing sentences"?

—*Gian-carlo Rota*

In preface to "*The Mathematical Experience*". Philip J. Davis and Reuben Hersh. Boston: Birkhäuser, 1981.