

Accurate Eigenvalues for Fast Trains. Participants from Berlin, Bologna and Basel arrived by train, and so did those who had flown into Frankfurt or Düsseldorf from Madrid, Manchester and Raleigh/Durham. The occasion: The IWASEP5¹ and GAMM² workshops in Hagen, Germany, during the week of 28 June 2004. The subject: accurate solution of eigenvalue problems, and linear algebra in systems & control theory. An application: trains. To be specific:

Vibration Analysis of Rails Excited by High-Speed Trains. This was the topic of talks by Volker Mehrmann and Christian Mehl from the Technical University of Berlin. In collaboration with the company SFE they study the resonances when rail tracks are excited by high speed trains, the goal being to reduce noise and vibrations in the trains. The new ICE trains travel across Europe at speeds as high as 300 km/h but the numerical methods used to design them are at least 30 years old. More often than not the classical finite element packages produce answers that are plain wrong, failing to deliver even a single correct digit. Volker Mehrmann and Christian Mehl showed how modern methods for linear algebra can provide answers accurate to 3 digits in single (!) precision, without a change in the finite element model. The idea is to carefully exploit structure in the eigenvalue problem.

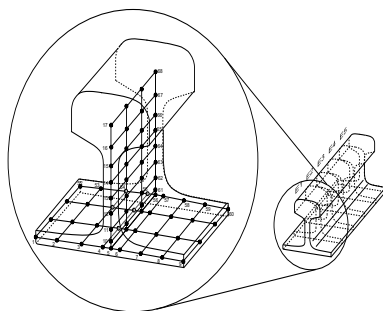


FIG. 0.1. *Discretization of the Rail.*

To understand what Volker Mehrmann, Christian Mehl and their collaborators have done, let's start with the modelling of the rail. If the rail is straight and infinitely long, a simple finite element discretization, as shown in Figure 0.1, produces an infinite-dimensional system of ordinary differential equations

$$M\ddot{x} + D\dot{x} + Kx = F,$$

where the matrices M , D and K are block-tridiagonal. If, in addition, the rail sections between cross ties are identical, then the system is also periodic. After Fourier transforming, and combining into one vector y_j all unknowns located between cross ties j and $j + 1$ we get a 3-term difference equation with constant coefficients,

$$A_1^T y_{j-1} + A_0 y_j + A_1 y_{j+1} = F_j,$$

where the complex coefficient matrices depend on the excitation frequency (here the superscript T denotes the transpose). The matrix A_0 is symmetric ($A_0^T = A_0$) while

¹<http://www.fernuni-hagen.de/mathphys/iwasep5/>

²<http://www.math.tu-berlin.de/~kressner/gamm04/>

the matrix A_1 is singular. The ansatz $y_{j+1} = \kappa y_j$ leads to a rational eigenvalue problem

$$R(\kappa)y = 0, \quad \text{where} \quad R(\kappa) \equiv \frac{1}{\kappa}A_1^T + A_0 + \kappa A_1.$$

Because $R(\kappa) = R(\frac{1}{\kappa})^T$, this eigenvalue problem is palindromic (in imitation of real-world palindromes such as: Was it a car or a cat I saw?) As a result the eigenvalues occur in pairs $(\kappa, 1/\kappa)$. Such a spectrum is called symplectic; an example is shown in Figure 0.2. To analyse the vibrations in the tracks, one needs to compute all finite, non-zero eigenvalues and eigenvectors for many frequencies in the range 0-5000 Hz.

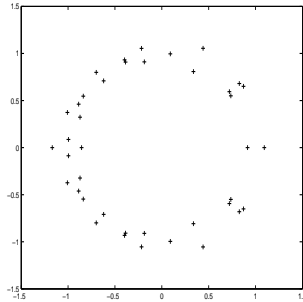


FIG. 0.2. *Example of a Symplectic Spectrum.*

Eigenvalues. A popular approach for solving the rational eigenvalue problem $R(\kappa)y = 0$ is to convert it to a polynomial eigenvalue problem which then in turn is linearized. To this end write $R(\kappa)y = 0$ as the polynomial eigenvalue problem

$$P(\lambda)y = 0, \quad \text{where} \quad P(\lambda) = \lambda^2 A_1^T + \lambda A_0 + A_1.$$

Then apply a classical linearization $z = \lambda y$ to get something like

$$\begin{pmatrix} 0 & I \\ -A_1 & -A_0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \lambda \begin{pmatrix} I & 0 \\ 0 & A_1^T \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}.$$

This is a generalized eigenvalue problem, which can be solved by public domain software, such as the MATLAB function *eig*. That's it, problem solved!

Not really. All we have done is produce garbage. The two big matrices in the linearized problem are not symmetric; the structure of the original problem is destroyed. This means numerical methods are not going to deliver a symplectic spectrum, i.e. the computed eigenvalues are not going to exhibit mirror symmetry, and computed eigenvalues at 0 and ∞ are not going to occur in pairs. We cannot trust any of the computed eigenvalues. The conventional methods have failed because they cannot see the structure in the original problem and end up producing inaccurate, useless eigenvalues.

What to do? If we want computed eigenvalues that are accurate then, at the very least, they should retain the mirror symmetry of a symplectic spectrum. To get a chance at delivering a symplectic spectrum, a numerical method is much better off if it can work on a palindromic polynomial. This means, we need to linearize without losing the palindromic structure. To this end, Volker Mehrmann and Christian Mehl,

together with Niloufer and Steve Mackey, have developed a theory of structure preserving linearizations, which yields a whole vector space full of linearizations. Not all of the linearizations are useful, and choosing a 'best' one (whatever that means) is an open problem. For the application at hand, this one works:

$$(\lambda Z + Z^T) \begin{pmatrix} z \\ y \end{pmatrix} = 0, \quad \text{where } Z \equiv \begin{pmatrix} A_1^T & A_0 - A_1 \\ A_1^T & A_1^T \end{pmatrix}$$

is a palindromic linearization³ of $P(\lambda)$.

We are not home free, yet. As it turns out the computed eigenvalues in this problem are badly scaled: their magnitudes range from 10^{-15} to 10^{15} . That's because, when we look at the exact eigenvalues of $P(\lambda)$, many are located at 0 and ∞ . It is therefore imperative to perform a similarity transformation that removes (deflates) the eigenvalues at 0 and ∞ , while managing to preserve the palindromic structure. Only then can the remaining eigenvalues of $P(\lambda)$ be computed with a carefully customized Jacobi method.

The Upshot. It pays to preserve structure, if it can be done in a numerically viable fashion. In the above train problem, structure-preserving linear algebra methods rescued an otherwise moribund computation. They made it possible to compute accurate answers – in the face of a simplistic computational model and a coarse discretization.

Many details have been swept under the rug in the preceding description. They can be found in the following references, which represent just a few of the many papers on structure-preserving methods in linear algebra. A general survey of quadratic eigenvalue problems, including the conventional linearizations, is given in [2]. A first attempt at structure preserving linearizations for matrix polynomials was made in [1]. Two forthcoming papers introduce vector spaces of linearizations for matrix polynomials [3], and linearizations for palindromic polynomials [4]. The work on the SFE project is described in the Master's thesis [5].

REFERENCES

- [1] V. Mehrmann and D. Watkins. Polynomial eigenvalue problems with Hamiltonian structure. *Electr. Trans. Num. Anal.*, 13:106–113, 2002.
- [2] F. Tisseur and K. Meerbergen. A survey of the quadratic eigenvalue problem. *SIAM Rev.*, 43:234–286, 2001.
- [3] D.S. Mackey, N. Mackey, C. Mehl and V. Mehrmann, Linearization Spaces for Matrix Polynomials. In preparation.
- [4] D.S. Mackey, N. Mackey, C. Mehl and V. Mehrmann, Palindromic Polynomial Eigenvalue Problems: Good vibrations from good linearizations. In preparation.
- [5] Andreas Hilliges, Numerische Lösung von quadratischen Eigenwertproblemen mit Anwendungen in der Schienendynamik, Master's Thesis, Technical University Berlin, Germany, July 2004

³Strictly speaking, it's only a linearization if -1 is not an eigenvalue. If it is, it must be removed (deflated) first.