The Linear Algebra Aspects of PageRank

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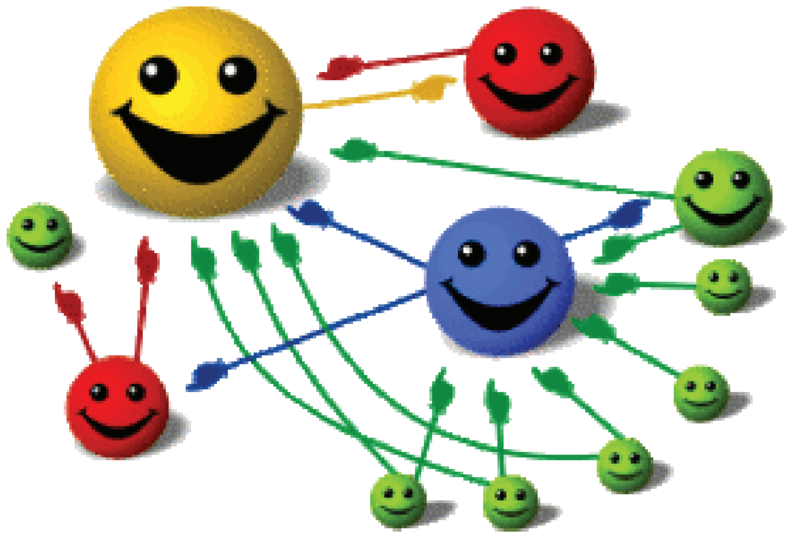
Thanks to Teresa Selee and Rebecca Wills

More PageRank More Visitors



Two Factors

Determine where Google displays a web page on the *Search Engine Results Page*:

1. PageRank (links)

   A page has high PageRank
   if many pages with high PageRank link to it

2. Hypertext Analysis (page contents)

   Text, fonts, subdivisions, location of words, contents of neighbouring pages

PageRank

An objective measure of the citation importance of a web page [Brin & Page 1998]

- Assigns a rank to every web page
- Influences the order in which Google displays search results
- Based on link structure of the web graph
- Does not depend on contents of web pages
- Does not depend on query

PageRank

. . . continues to provide the basis for all of our web search tools http://www.google.com/technology/

- “Links are the currency of the web”
- Exchanging & buying of links
- BO (backlink obsession)
- Search engine optimization

Overview

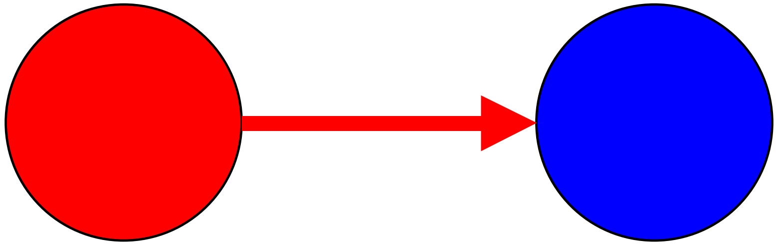
- Mathematical Model of Internet
- Computation of PageRank
- Sensitivity of PageRank to Rounding Errors
- Addition & Deletion of Links
- Web Pages that have no Outlinks
- Is the Ranking Correct?

Mathematical Model of Internet

1. Represent internet as graph
2. Represent graph as stochastic matrix
3. Make stochastic matrix more convenient
   $\Longrightarrow$ Google matrix
4. dominant eigenvector of Google matrix
   $\Longrightarrow$ PageRank

The Internet as a Graph

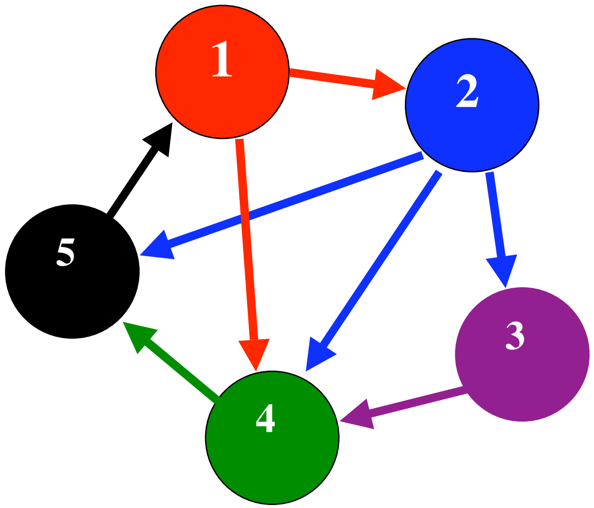
Link from one web page to another web page



Web graph:

Web pages = nodes
Links = edges

The Web Graph as a Matrix

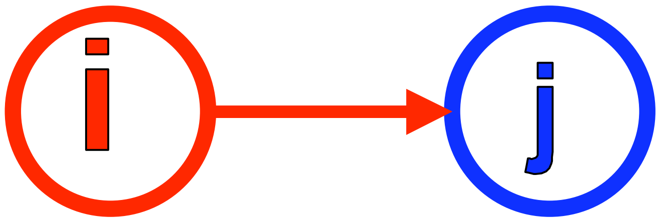


$$S = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Links = nonzero elements in matrix

Elements of Matrix S

Assume: every page i has $l_i \geq 1$ outlinks



If page i has link to page j then $s_{ij} = 1/l_i$
else $s_{ij} = 0$

Probability that surfer moves from page i to page j

Properties of Matrix S

- Stochastic: $0 \leq s_{ij} \leq 1 \quad S\mathbb{1} = \mathbb{1}$
- Dominant left eigenvector:

$$\omega^T S = \omega^T \qquad \omega \geq 0 \qquad \|\omega\|_1 = 1$$

- ω_i is probability that surfer visits page i

But: ω not unique
if S has several eigenvalues equal to 1

Remedy: Make the matrix more convenient

Google Matrix

Convex combination

$$G = \alpha S + \underbrace{(1-\alpha)\mathbb{1}v^T}_{\text{rank } 1}$$

- Stochastic matrix S
- Damping factor $0 \le \alpha < 1$
  e.g. $\alpha = .85$
- Column vector of all ones $\mathbb{1}$
- Personalization vector $v \ge 0 \quad \|v\|_1 = 1$
  Models teleportation

Properties of Google Matrix G

$$G = \alpha S + (1-\alpha)\mathbb{1}v^T$$

- Stochastic, reducible
- Eigenvalues of G:

$$1 > \alpha\lambda_2(S) \geq \alpha\lambda_3(S) \geq \dots$$

- Unique dominant left eigenvector:

$$\pi^T G = \pi^T \qquad \pi \geq 0 \qquad \|\pi\|_1 = 1$$

PageRank

Google Matrix

$$G = \underbrace{\alpha S}_{\text{Links}} + \underbrace{(1-\alpha)\mathbb{1}v^T}_{\text{Personalization}}$$

$$\pi^T G = \pi^T \qquad \pi \geq 0 \qquad \|\pi\|_1 = 1$$

π_i is PageRank of web page i

PageRank $\doteq$ dominant left eigenvector of G

How Google Ranks Web Pages

- Model:
  Internet $\longrightarrow$ web graph $\longrightarrow$ stochastic matrix G
- Computation:
  PageRank π is eigenvector of G

  π_i is PageRank of page i
- Display:

  If $\pi_i > \pi_k$ then
  page i may* be displayed before page k

* depending on hypertext analysis

History

- The anatomy of a large-scale hypertextual web search engine
  Brin & Page 1998
- US patent for PageRank granted in 2001
- Eigenstructure of the Google Matrix
  Haveliwala & Kamvar 2003
  Eldén 2003
  Serra-Capizzano 2005

Statistics

- Google indexes 10s of billions of web pages
- “3 times more than any competitor”
- Google serves $\geq \mathbf{200}$ million queries per day
- Each query processed by $\geq \mathbf{1000}$ machines
- All search engines combined serve a total of $\geq \mathbf{500}$ million queries per day

[Desikan, 26 October 2006]

Computation of PageRank

The world's largest matrix computation
[Moler 2002]

- Eigenvector
- Matrix dimension is 10s of billions
- The matrix changes often
  250,000 new domain names every day
- Fortunately: Matrix is sparse

Power Method

Want: π such that $\quad \pi^T G = \pi^T$

Power method:

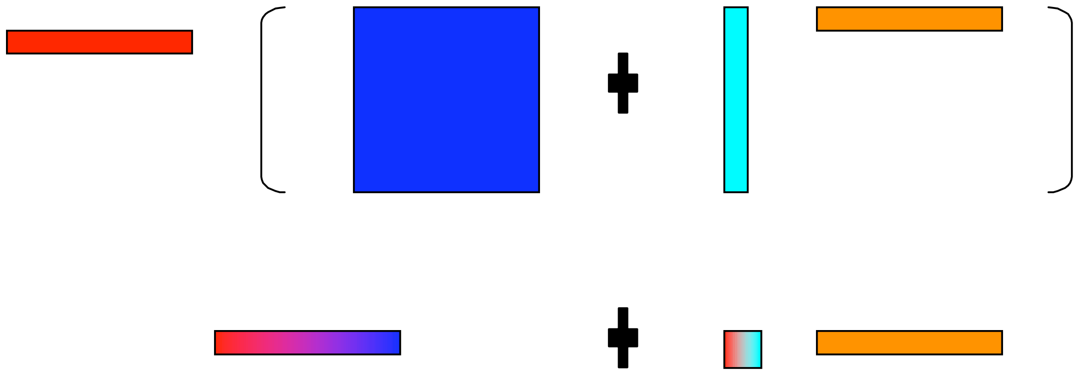
Pick an initial guess $x^{(0)}$
Repeat

$$[x^{(k+1)}]^T := [x^{(k)}]^T G$$

Each iteration is a matrix vector multiply

Matrix Vector Multiply

$$x^T G = \quad x^T \left[\alpha S + (1-\alpha)\mathbb{1} v^T\right]$$



An Iteration is Cheap

Google matrix $G = \alpha S + (1-\alpha)\mathbb{1} v^T$

Vector $x \geq 0 \quad \|x\|_1 = 1$

$$\begin{aligned} x^T G &= x^T \left[\alpha S + (1-\alpha)\mathbb{1} v^T\right] \\ &= \alpha\, x^T S + (1-\alpha)\ \underbrace{x^T \mathbb{1}}_{=1}\ v^T \\ &= \alpha\, x^T S + (1-\alpha) v^T \end{aligned}$$

Cost: # non-zero elements in S

Error in Power Method

$$\pi^T G = \pi^T \qquad G = \alpha S + (1-\alpha)\mathbb{1} v^T$$

$$\begin{aligned}
[x^{(k+1)} - \pi]^T &= [x^{(k)}]^T G - \pi^T G \\
&= \alpha [x^{(k)}]^T S - \alpha \pi^T S \\
&= \alpha [x^{(k)} - \pi]^T S
\end{aligned}$$

$$\underbrace{\|x^{(k+1)} - \pi\|}_{\text{iteration } k+1} \leq \alpha \underbrace{\|x^{(k)} - \pi\|}_{\text{iteration } k}$$

Norms: 1, ∞

Error in Power Method

$$\pi^T G = \pi^T \qquad G = \alpha S + (1-\alpha)\mathbb{1} v^T$$

Error after k iterations:

$$\|x^{(k)} - \pi\| \leq \alpha^k \underbrace{\|x^{(0)} - \pi\|}_{\leq 2}$$

Norms: $1, \infty$ [Bianchini, Gori & Scarselli 2003]

Error bound does not depend on matrix dimension

Iteration Counts for Different α

bound: k such that $2\,\alpha^k \le 10^{-8}$

Termination based on residual norms vs bound

α	$n = 281903$	$n = 683446$	bound
.85	69	65	119
.90	107	102	166
.95	219	220	415
.99	1114	1208	2075

Fewer iterations than predicted by bound

Advantages of Power Method

- Converges to unique vector
- Convergence rate α
- Convergence independent of matrix dimension
- Vectorizes
- Storage for only a single vector
- Sparse matrix operations
- Accurate (no subtractions)
- Simple (few decisions)

But: can be slow

PageRank Computation

- Power method

  Page, Brin, Motwani & Winograd 1999

  Bianchini, Gori & Scarselli 2003

- Acceleration of power method

  Kamvar, Haveliwala, Manning & Golub 2003

  Haveliwala, Kamvar, Klein, Manning & Golub 2003

  Brezinski & Redivo-Zaglia 2004, 2006

  Brezinski, Redivo-Zaglia & Serra-Capizzano 2005

- Aggregation/Disaggregation

  Langville & Meyer 2002, 2003, 2006

  Ipsen & Kirkland 2006

PageRank Computation

- Methods that adapt to web graph

  Broder, Lempel, Maghoul & Pedersen 2004 Kamvar, Haveliwala & Golub 2004

  Haveliwala, Kamvar, Manning & Golub 2003

  Lee, Golub & Zenios 2003

  Lu, Zhang, Xi, Chen, Liu, Lyu & Ma 2004
  Ipsen & Selee 2006

- Krylov methods

  Golub & Greif 2004
  Del Corso, Gullí, Romani 2006

PageRank Computation

- Schwarz & asynchronous methods

  Bru, Pedroche & Szyld 2005

  Kollias, Gallopoulos & Szyld 2006

- Linear system solution

  Arasu, Novak, Tomkins & Tomlin 2002

  Arasu, Novak & Tomkins 2003

  Bianchini, Gori & Scarselli 2003

  Gleich, Zukov & Berkin 2004

  Del Corso, Gullí & Romani 2004

  Langville & Meyer 2006

PageRank Computation

- Surveys of numerical methods:
  Langville & Meyer 2004
  Berkhin 2005
  Langville & Meyer 2006 (book)

Sensitivity of PageRank

How sensitive is PageRank π to small perturbations, e.g. rounding errors

- Changes in matrix S
- Changes in damping factor α
- Changes in personalization vector v

Perturbation Theory

For Markov chains

Schweizer 1968, Meyer 1980

Haviv & van Heyden 1984

Funderlic & Meyer 1986

Golub & Meyer 1986

Seneta 1988, 1991

Ipsen & Meyer 1994

Kirkland, Neumann & Shader 1998

Cho & Meyer 2000, 2001

Kirkland 2003, 2004

Perturbation Theory

For Google matrix

Chien, Dwork, Kumar & Sivakumar 2001

Ng, Zheng & Jordan 2001

Bianchini, Gori & Scarselli 2003

Boldi, Santini & Vigna 2004, 2005

Langville & Meyer 2004

Golub & Greif 2004

Kirkland 2005, 2006

Chien, Dwork, Kumar, Simon & Sivakumar 2005

Avrechenkov & Litvak 2006

Changes in the Matrix S

Exact:

$$\pi^T G = \pi^T \qquad G = \alpha S + (1-\alpha)\mathbb{1} v^T$$

Perturbed:

$$\tilde{\pi}^T \tilde{G} = \tilde{\pi}^T \qquad \tilde{G} = \alpha(S + E) + (1-\alpha)\mathbb{1} v^T$$

Error:

$$\tilde{\pi}^T - \pi^T = \alpha \tilde{\pi}^T E (I - \alpha S)^{-1}$$

$$\|\tilde{\pi} - \pi\|_1 \leq \frac{\alpha}{1-\alpha} \|E\|_\infty$$

Changes in α and v

- Change in amplification factor:

$$\tilde{G} = (\alpha + \mu)S + (1 - (\alpha + \mu))\,\mathbb{1}v^T$$

Error: $\|\tilde{\pi} - \pi\|_1 \leq \frac{2}{1-\alpha}\,|\mu|$

[Langville & Meyer 2004]

- Change in personalization vector:

$$\tilde{G} = \alpha S + (1 - \alpha)\mathbb{1}(v + f)^T$$

Error: $\|\tilde{\pi} - \pi\|_1 \leq \|f\|_1$

Sensitivity of PageRank π

$$\pi^T G = \pi^T \qquad G = \alpha S + (1-\alpha)\mathbb{1} v^T$$

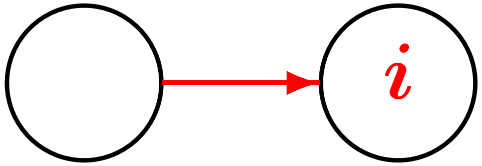
Changes in

- S: condition number $\alpha/(1-\alpha)$
- α: condition number $2/(1-\alpha)$
- v: condition number 1

$\alpha = .85$: condition numbers ≤ 14
$\alpha = .99$: condition numbers ≤ 200

PageRank insensitive to rounding errors

Adding an In-Link



$$\tilde{\pi}_i > \pi_i$$

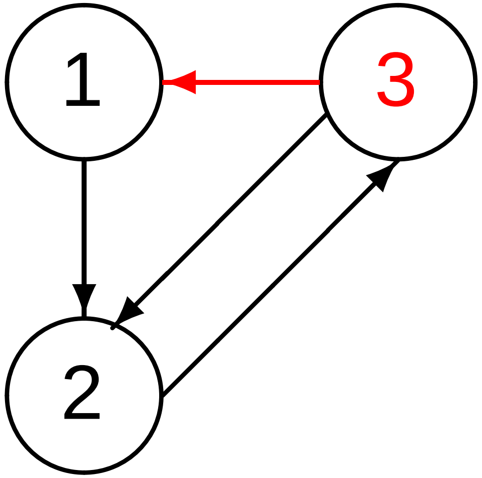
Adding an in-link increases PageRank (monotonicity)

Removing an in-link decreases PageRank

[Chien, Dwork, Kumar & Sivakumar 2001]

[Chien, Dwork, Kumar, Simon & Sivakumar 2005]

Adding an Out-Link



$$\tilde{\pi}_3 = \frac{1+\alpha+\alpha^2}{3(1+\alpha+\alpha^2/2)} < \pi_3 = \frac{1+\alpha+\alpha^2}{3(1+\alpha)}$$

Adding an out-link may decrease PageRank

Justification for TrustRank

Adjust personalization vector to combat web spam
[Gyöngyi, Garcia-Molina, Pedersen 2004]

Increase $\boldsymbol{v}$ for page $\boldsymbol{i}$: $\boldsymbol{v_i := v_i + \phi}$
Decrease $\boldsymbol{v}$ for page $\boldsymbol{j}$: $\boldsymbol{v_j := v_j - \phi}$

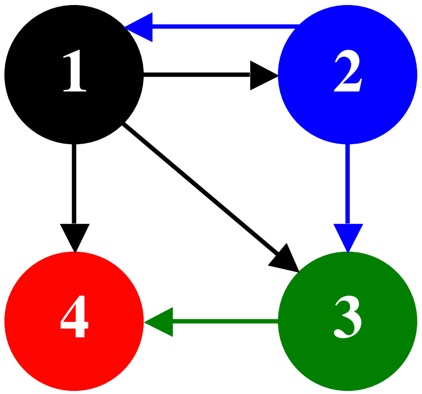
PageRank of page $\boldsymbol{i}$ increases: $\boldsymbol{\tilde{\pi}_i > \pi_i}$
PageRank of page $\boldsymbol{j}$ decreases: $\boldsymbol{\tilde{\pi}_j < \pi_j}$

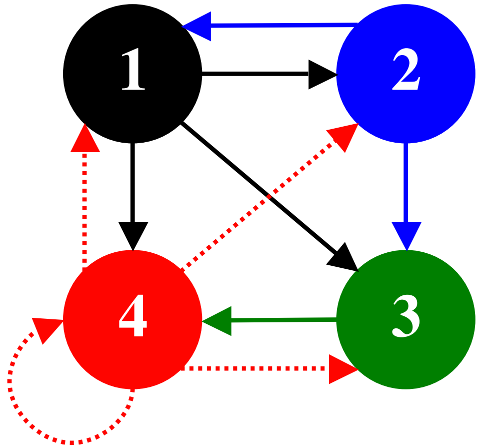
Total change in PageRank $\quad \boldsymbol{\|\tilde{\pi} - \pi\|_1 \leq 2\phi}$

Web Pages that have no Outlinks

- Technical term: Dangling Nodes
- Examples:
  Image files
  PDF and PS files
  Pages whose links have not yet been crawled
  Protected web pages
- 50%-80% of all web pages
- Problem: zero rows in matrix
- Popular fix: Insert artificial links

Dangling Node Fix





$$\begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ w_1 & w_2 & w_3 & w_4 \end{pmatrix}$$

Inside the Stochastic Matrix S

Number pages so that dangling nodes are last

$$S = \begin{pmatrix} H \\ 0 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ \mathbb{1} w^T \end{pmatrix}}_{\text{rank } 1}$$

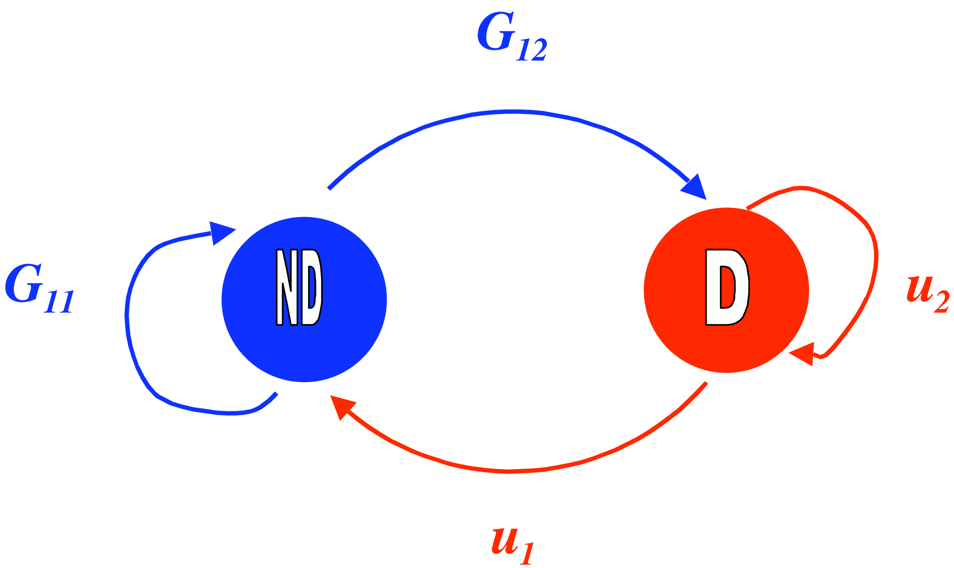
Links from nondangling nodes: H

Dangling node vector $w \geq 0 \quad \|w\|_1 = 1$

Google matrix $\quad G = \alpha \begin{pmatrix} H \\ \mathbb{1} w^T \end{pmatrix} + (1-\alpha)\mathbb{1} v^T$

Partitioning the Google Matrix

$$G = \begin{pmatrix} G_{11} & G_{12} \\ \mathbb{1} u_1^T & \mathbb{1} u_2^T \end{pmatrix}$$



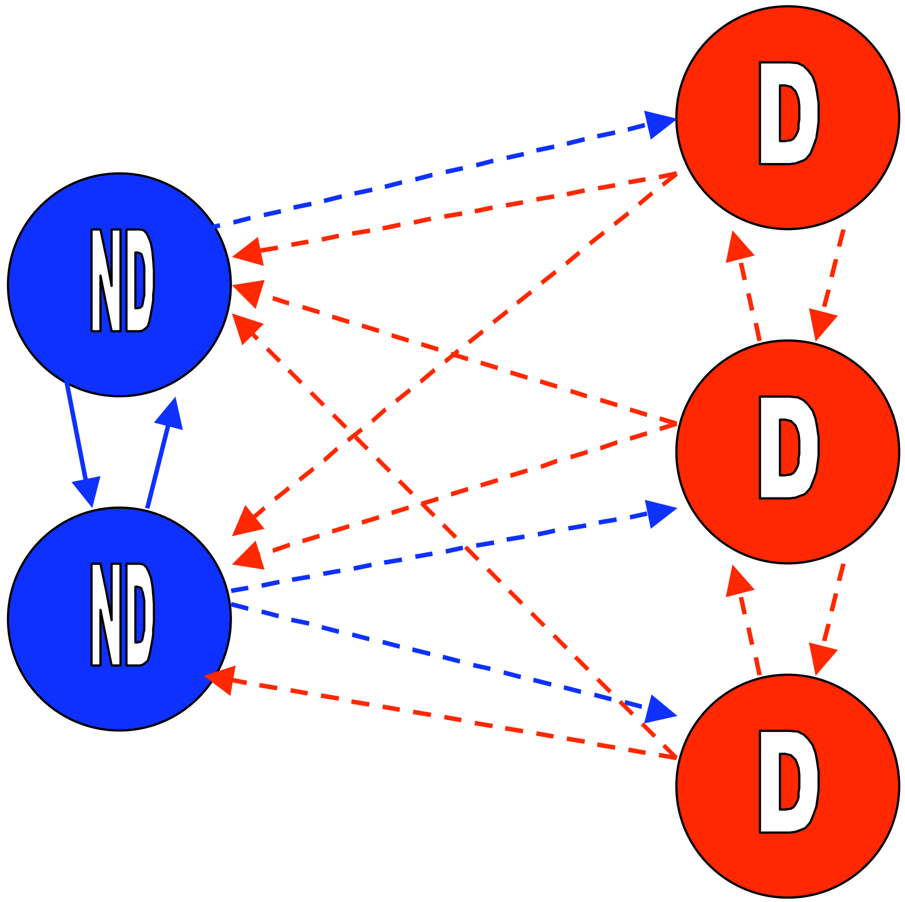
$$\begin{pmatrix} u_1^T & u_2^T \end{pmatrix} = \underbrace{\alpha w^T}_{\text{dangling nodes}} + \underbrace{(1-\alpha) v^T}_{\text{personalization}}$$

Lumping

Separate dangling and nondangling nodes
"Lump" all dangling nodes into single node

- Stochastic matrices:
  Kemeny & Snell 1960
  Dayar & Stewart 1997
  Jernigan & Baran 2003
  Gurvits & Ledoux 2005
- Google matrix:
  Lee, Golub & Zenios 2003
  Ipsen & Selee 2006

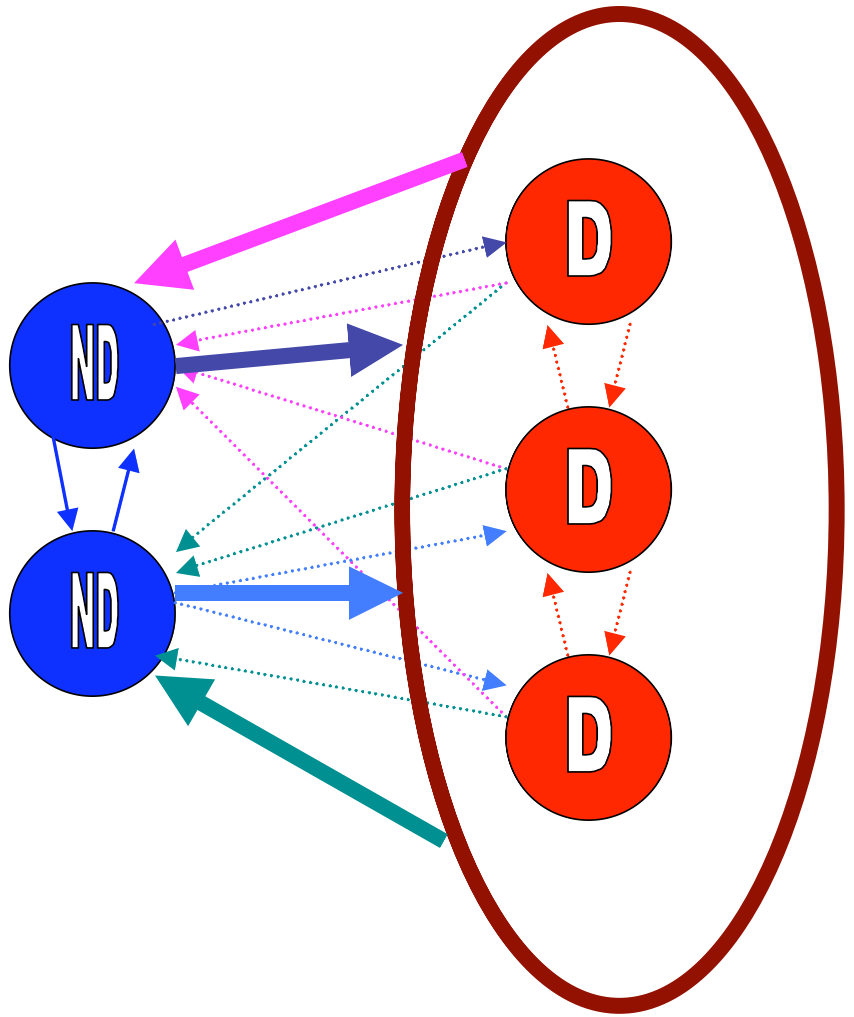
Example



⟶: real links

⟶: artificial links

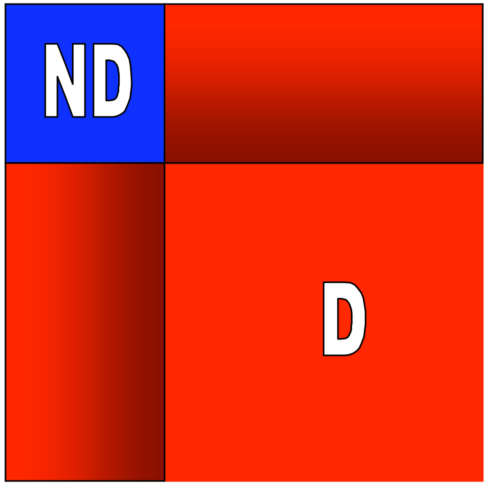
Lumped Example



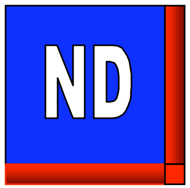
Google Lumping

1. “Lump” all dangling nodes into a single node
2. Compute dominant eigenvector of
   smaller, lumped matrix
   $\Longrightarrow$ PageRank of nondangling nodes
3. Determine PageRank of dangling nodes
   with one matrix vector multiply

1. Lump Dangling Nodes



Google Matrix G



Lumped matrix L

1. Lump Dangling Nodes

$$G = \begin{pmatrix} G_{11} & G_{12} \\ \mathbb{1} u_1^T & \mathbb{1} u_2^T \end{pmatrix}$$

Lump $n - d$ dangling nodes into a single node

$\Longrightarrow$ Lumped matrix has dimension $d + 1$

$$L = \begin{pmatrix} G_{11} & G_{12}\mathbb{1} \\ u_1^T & u_2^T \mathbb{1} \end{pmatrix}$$

Stochastic, same nonzero eigenvalues as G

2. Eigenvector of Lumped Matrix

$$\boldsymbol{L} = \begin{pmatrix} \boldsymbol{G_{11}} & \boldsymbol{G_{12}}\mathbb{1} \\ \boldsymbol{u_1^T} & \boldsymbol{u_2^T}\mathbb{1} \end{pmatrix}$$

Lumped matrix with $\boldsymbol{d}$ nondangling nodes

Compute eigenvector of lumped matrix

$$\boldsymbol{\sigma^T L = \sigma^T} \qquad \boldsymbol{\sigma \geq 0} \qquad \boldsymbol{\|\sigma\|_1 = 1}$$

PageRank of nondangling nodes: $\boldsymbol{\sigma_{1:d}}$

3. Dangling Nodes

$$G = \begin{pmatrix} G_{11} & G_{12} \\ \mathbb{1} u_1^T & \mathbb{1} u_2^T \end{pmatrix} \qquad L = \begin{pmatrix} G_{11} & G_{12}\mathbb{1} \\ u_1^T & u_2^T \mathbb{1} \end{pmatrix}$$

Eigenvector of lumped matrix: $\sigma^T L = \sigma^T$

PageRank of dangling nodes:

$$\sigma^T \begin{pmatrix} G_{12} \\ u_2^T \end{pmatrix}$$

One matrix vector multiply

Summary: Dangling Nodes

n web pages with $n-d$ dangling nodes

- PageRank $\sigma_{1:d}$ of d nondangling nodes: from lumped matrix L of dimension $d+1$
- PageRank of dangling nodes: one matrix vector multiply
- Total PageRank

$$\pi^T = \begin{pmatrix} \underbrace{\sigma_{1:d}^T}_{\text{nondangling}} & \underbrace{\sigma^T \begin{pmatrix} G_{12} \\ u_2^T \end{pmatrix}}_{\text{dangling}} \end{pmatrix}$$

Summary: Dangling Nodes, ctd.

- PageRank of nondangling nodes is independent of PageRank of dangling nodes
- PageRank of nondangling nodes can be computed separately
- Power method on lumped matrix $\boldsymbol{L}$:

  same convergence rate as for $\boldsymbol{G}$

  but $\boldsymbol{L}$ much smaller than $\boldsymbol{G}$

  speed increases with # dangling nodes

Is the Ranking Correct?

$$\pi^T = \begin{pmatrix} .23 & .24 & .26 & .27 \end{pmatrix}$$

- $[x^{(k)}]^T = \begin{pmatrix} .27 & .26 & .24 & .23 \end{pmatrix}$

  $\|x^{(k)} - \pi\|_\infty = .04$

  Small error, but incorrect ranking

- $[x^{(k)}]^T = \begin{pmatrix} 0 & .001 & .002 & .997 \end{pmatrix}$

  $\|x^{(k)} - \pi\|_\infty = .727$

  Large error, but correct ranking

Is the Ranking Correct?

After k iterations of power method:

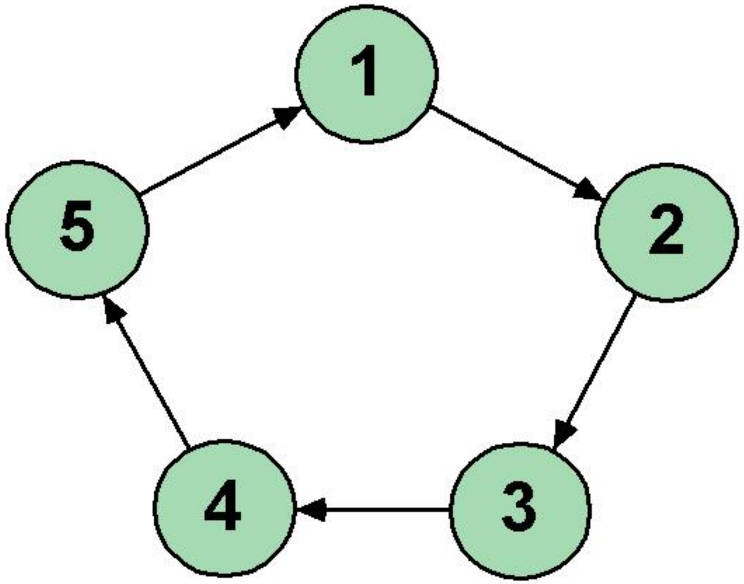
Error: $\|x^{(k)} - \pi\| \leq 2\,\alpha^k$

But: Do the components of $x^{(k)}$ have the same ranking as those of π?

Rank-stability, rank-similarity: [Lempel & Moran, 2005]

[Borodin, Roberts, Rosenthal & Tsaparas 2005]

Web Graph is a Ring



$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

[Ipsen & Wills]

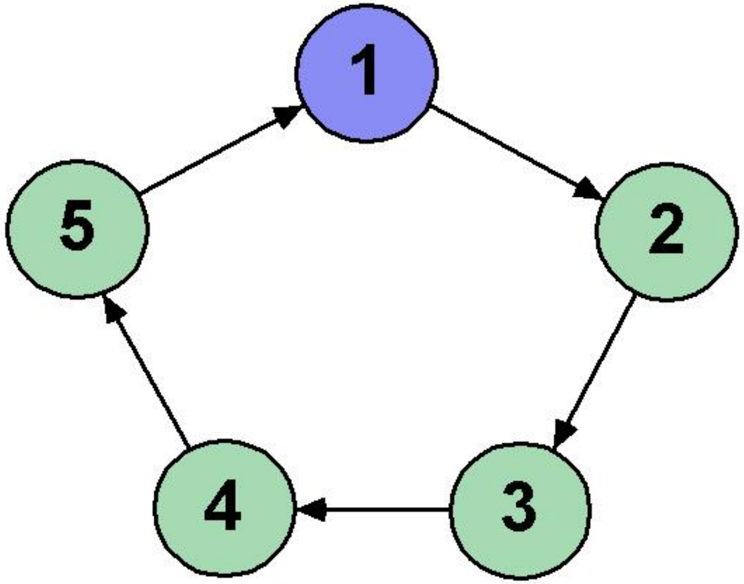
All Pages are Trusted

S is circulant of order n, $\quad v = \frac{1}{n}\mathbb{1}$

- PageRank: $\quad \pi = \frac{1}{n}\mathbb{1}$
  All pages have same PageRank

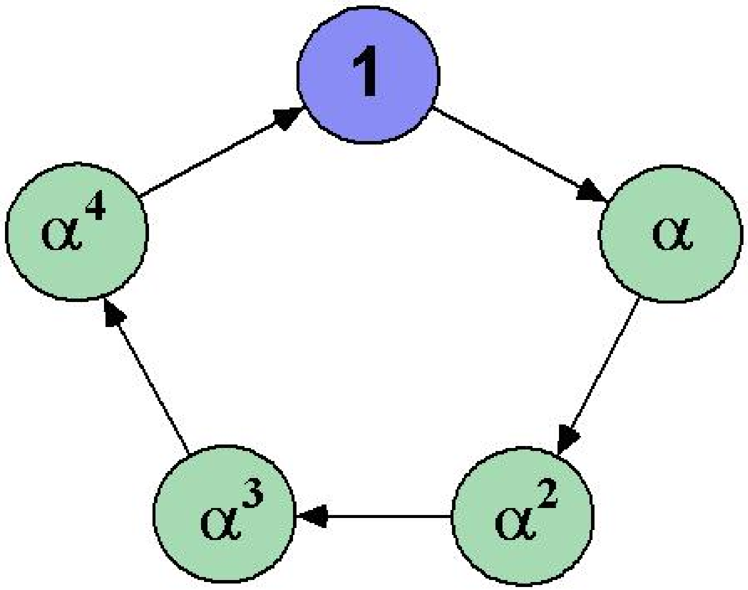
- Power method
  $x^{(0)} = v$: $\quad x^{(0)} = \pi \qquad$ correct ranking
  $x^{(0)} \neq v$: $\quad [x^{(k)}]^T \sim \frac{1}{n}\mathbb{1}^T + \alpha^k \left([x^{(0)}]^T S^k - \frac{1}{n}\mathbb{1}\right)$
  Ranking does not converge (in exact arithmetic)

Only One Page is Trusted



$$v^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Only One Page is Trusted



$$\pi^T \sim \begin{pmatrix} 1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 \end{pmatrix}$$

PageRank decreases with distance from page 1

Only One Page is Trusted

S is circulant of order n, $\quad v = e_1$

- PageRank: $\quad \pi^T \sim \begin{pmatrix} 1 & \alpha & \dots & \alpha^{n-1} \end{pmatrix}$

- Power method with $x^{(0)} = v$:

  $[x^{(k)}]^T \sim \begin{pmatrix} 1 & \alpha & \dots & \alpha^{k-1} & \frac{\alpha^k}{1-\alpha} & 0 & \dots & 0 \end{pmatrix}$

  $[x^{(n)}]^T \sim \begin{pmatrix} 1 + \frac{\alpha^n}{1-\alpha} & \alpha & \alpha^2 & \dots & \alpha^{n-1} \end{pmatrix}$

  Rank convergence in n iterations

Too Many Iterations

Power method with $x^{(0)} = v = e_1$:

- After n iterations:
  $[x^{(n)}]^T \sim \left(1 + \frac{\alpha^n}{1-\alpha} \quad \alpha \quad \alpha^2 \quad \ldots \quad \alpha^{n-1}\right)$
- After $n + 1$ iterations:
  $[x^{(n+1)}]^T \sim \left(1 + \alpha^n \quad \alpha + \frac{\alpha^{n+1}}{1-\alpha} \quad \alpha^2 \quad \ldots \quad \alpha^{n-1}\right)$

  If $\alpha = .85$, $n = 10$: $\quad \alpha + \frac{\alpha^{n+1}}{1-\alpha} > 1 + \alpha^n$

  Additional iterations can destroy a converged ranking

Recovery of Ranking

S is circulant of order n

- After k iterations:

$$[x^{(k)}]^T = \alpha^k \, [x^{(0)}]^T S^k + (1-\alpha) v^T \sum_{j=0}^{k-1} \alpha^j S^j$$

- After $k+n$ iterations:

$$[x^{(k+n)}]^T = \alpha^n \, [x^{(k)}]^T + (1-\alpha^n) \pi^T$$

If $x^{(k)}$ has correct ranking, so does $x^{(k+n)}$

Any Personalization Vector

S is circulant of order n

- PageRank: $\pi^T \sim v^T \sum_{j=0}^{n-1} \alpha^j S^j$
- Power method with $x^{(0)} = \frac{1}{n}\mathbb{1}$

$$[x^{(n)}]^T = \underbrace{(1-\alpha^n)}_{\text{scalar}}\pi^T + \underbrace{\frac{\alpha^n}{n}}_{\text{constant vector}}\mathbb{1}^T$$

For any v: rank convergence after n iterations

Problems with Ranking

- Ranking may never converge
- Additional iterations can destroy ranking
- Small error does not imply correct ranking
- Rank convergence depends on:
  α, v, initial guess, matrix dimension,
  structure of web graph
- How do we know when the ranking is correct?
- Even if successive iterates have the same ranking, their ranking may not be correct

Summary

- Google orders web pages according to: PageRank and hypertext analysis
- PageRank = left eigenvector of G

$$G = \alpha S + (1 - \alpha)\mathbb{1}v^T$$

- Power method: simple and robust
- Error in iteration k bounded by α^k
- Convergence rate largely independent of dimension and eigenvalues of G

Summary, ctd

- PageRank insensitive to rounding errors
- Adding in-links increases PageRank
- Adding out-links may decrease PageRank
- Dangling nodes = pages w/o outlinks
  Rank one change to hyperlink matrix
- Lumping:
  PageRank of nondangling nodes computed separately from PageRank of dangling nodes
- Ranking problem: DIFFICULT

User-Friendly Resources

- Rebecca Wills:
  Google's PageRank: The Math Behind the Search Engine
  Mathematical Intelligencer, 2006
- Amy Langville & Carl Meyer:
  Google's PageRank and Beyond
  The Science of Search Engine Rankings
  Princeton University Press, 2006
- Amy Langville & Carl Meyer:
  Broadcast of On-Air Interview, November 2006
  Carl Meyer's web page