Perturbation Bounds for Determinants and Characteristic Polynomials

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Characteristic Polynomials

$n \times n$ complex matrix A

$$\det(\lambda I - A) = \lambda^n + c_1 \lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n$$

$$c_1 = -\,\mathrm{trace}(A) \qquad c_n = (-1)^n \det(A)$$

Application (Dean Lee, NCSU)

Thermodynamic properties of systems of fermions
$Z_k \equiv (-1)^k c_k$ is partition function

Perturbation Bounds

$n \times n$ complex matrices A and $A + E$

$$\begin{aligned} \det(\lambda I - A) &= \lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n \\ \det(\lambda I - (A+E)) &= \lambda^n + \tilde{c}_1\lambda^{n-1} + \cdots + \tilde{c}_{n-1}\lambda + \tilde{c}_n \end{aligned}$$

- **Trace:**

$$|\tilde{c}_1 - c_1| = |\,\mathrm{trace}(A+E) - \mathrm{trace}(A)| = |\,\mathrm{trace}(E)| \leq n\|E\|_2$$

- **Determinant:**

$$|\tilde{c}_n - c_n| = |\det(A+E) - \det(A)| \leq ?$$

- **Other coefficients:**

  Bound $|\tilde{c}_k - c_k|$ by means of results for determinants

Existing Bounds for Determinants

- **Friedland 1982, Bhatia 1987**

  $$|\det(A+E)-\det(A)| \leq n \max\{\|A\|_p, \|A+E\|_p\}^{n-1} \|E\|_p$$

  Proof: Fréchet derivatives of wedge products

- **Godunov, Antonov, Kiriljuk & Kostin 1988**

  If $n\|A^{-1}\|_2\|E\|_2 < 1$ then

  $$\frac{|\det(A+E)-\det(A)|}{|\det(A)|} \leq \frac{n\|A^{-1}\|_2}{1-n\|A^{-1}\|_2\|E\|_2} \|E\|_2$$

- **Our idea:**

  Use determinant expansions of diagonal matrices

Determinant Expansion of Diagonal Matrices

$$\mathbf{A} = \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix} \qquad \mathbf{E} = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix}$$

Expansion:

$$\det(\mathbf{A} + \mathbf{E}) = \det(\mathbf{A}) + S_1 + S_2 + \det(\mathbf{E})$$

where

$$S_1 = \sigma_1 \det \begin{pmatrix} e_{22} & e_{23} \\ e_{32} & e_{33} \end{pmatrix} + \sigma_2 \det \begin{pmatrix} e_{11} & e_{13} \\ e_{31} & e_{33} \end{pmatrix} + \sigma_3 \det \begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix}$$

$$S_2 = \sigma_1\sigma_2\, e_{33} + \sigma_1\sigma_3\, e_{22} + \sigma_2\sigma_3\, e_{11}$$

Determinant Expansion of Diagonal Matrices

$$A = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix}$$

Expansion:

$$\det(A + E) = \det(A) + S_1 + \cdots + S_{n-1} + \det(E)$$

where

$$S_k = \sum_{1 \le i_1 < \cdots < i_k \le n} \sigma_{i_1} \cdots \sigma_{i_k} \det(E_{i_1 \ldots i_k})$$

$\sigma_{i_1} \cdots \sigma_{i_k}$	product of k diagonal elements of A
$\det(E_{i_1 \ldots i_k})$	principal minor of order $n - k$ of E

Determinant Bounds for Diagonal Matrices

$$A = \operatorname{diag}\begin{pmatrix}\sigma_1 & \dots & \sigma_n\end{pmatrix} \qquad \sigma_1 \geq \dots \geq \sigma_n \geq 0$$

$$|\det(A+E) - \det(A)| \leq |S_1| + \cdots + |S_{n-1}| + |\det(E)|$$

where

$$|S_k| \leq \sum_{1 \leq i_1 < \cdots < i_k \leq n} \sigma_{i_1} \cdots \sigma_{i_k} \, |\det(E_{i_1 \dots i_k})|$$

Hadamard's inequality:

$$|\det(E_{i_1 \dots i_k})| \leq \|E_{i_1 \dots i_k}\|^{n-k} \leq \|E\|^{n-k} \qquad (2 \text{ norm})$$

Elementary Symmetric Functions

$$|S_k| \leq \sum_{1\leq i_1<\cdots<i_k\leq n} \sigma_{i_1}\cdots\sigma_{i_k}\,\|E\|^{n-k}$$

kth elementary symmetric function of $\sigma_1, \ldots, \sigma_n$

$$s_k \equiv \sum_{1\leq i_1<\cdots<i_k\leq n} \sigma_{i_1}\cdots\sigma_{i_k}$$

If $n = 3$ then

$$\begin{aligned} s_1 &= \sigma_1 + \sigma_2 + \sigma_3 \\ s_2 &= \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_1 + \sigma_2\sigma_3 + \sigma_3\sigma_1 + \sigma_3\sigma_2 \\ s_3 &= \sigma_1\sigma_2\sigma_3 \end{aligned}$$

$$|S_k| \leq s_k\|E\|^{n-k}$$

Determinant Bound for Diagonal Matrices

- **Non negative diagonal matrix**

$$A = \mathrm{diag}\begin{pmatrix}\sigma_1 & \ldots & \sigma_n\end{pmatrix} \qquad \sigma_1 \geq \ldots \geq \sigma_n \geq 0$$

- **Elementary symmetric functions of $\sigma_1, \ldots, \sigma_n$**

$$s_k \equiv \sum_{1 \leq i_1 < \cdots < i_k \leq n} \sigma_{i_1} \cdots \sigma_{i_k}$$

- **Determinant perturbation bound**

$$|\det(A) - \det(A+E)| \leq s_{n-1}\|E\| + \cdots + s_1\|E\|^{n-1} + \|E\|^n$$

- **First order bound**

$$|\det(A) - \det(A+E)| \leq s_{n-1}\|E\| + \mathcal{O}\left(\|E\|^2\right)$$

where $s_{n-1} \leq n\,\sigma_1 \ldots \sigma_{n-1}$

Determinant Bound for General Matrices

- Complex $n \times n$ matrix A with singular values

$$\sigma_1 \geq \ldots \geq \sigma_n \geq 0$$

- Elementary symmetric functions of singular values

$$s_k \equiv \sum_{1 \leq i_1 < \cdots < i_k \leq n} \sigma_{i_1} \cdots \sigma_{i_k}$$

- Determinant perturbation bound

$$|\det(A) - \det(A+E)| \leq s_{n-1}\|E\| + \cdots + s_1\|E\|^{n-1} + \|E\|^n$$

- First order bound

$$|\det(A) - \det(A+E)| \leq s_{n-1}\|E\| + \mathcal{O}\left(\|E\|^2\right)$$

where $s_{n-1} \leq n\,\sigma_1 \ldots \sigma_{n-1}$

Derivative of Determinant

If x is a real scalar then

$$\left| \frac{d}{dx} \det(A + xE)|_{x=0} \right| \leq s_{n-1} \, \|E\|$$

where $s_{n-1} \leq n \, \sigma_1 \ldots \sigma_{n-1}$

Local (absolute) condition number for determinant:
$(n-1)$st elementary symmetric function of singular values

Relative Bound

- If A is nonsingular then

$$1/\det(A) = \det(A^{-1})$$

- Determinant expansion of $I + A^{-1}E$

$$\frac{\det(A+E) - \det(A)}{\det(A)} = S_1 + \cdots + S_{n-1} + \det(A^{-1}E)$$

- Minors

$$S_k \equiv \sum_{1 \le i_1 < \cdots < i_k \le n} \det((A^{-1}E)_{i_1 \ldots i_k})$$

- Relative perturbation bound

$$\frac{|\det(A+E) - \det(A)|}{|\det(A)|} \le \left(\|A^{-1}\| \, \|E\| + 1 \right)^n - 1$$

Coefficients of Characteristic Polynomial

$n \times n$ complex matrices A and $A + E$

$$\begin{aligned} \det(\lambda I - A) &= \lambda^n + c_1 \lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n \\ \det(\lambda I - (A+E)) &= \lambda^n + \tilde{c}_1 \lambda^{n-1} + \cdots + \tilde{c}_{n-1}\lambda + \tilde{c}_n \end{aligned}$$

Friedland 1982, Bhatia 1987

$$|\tilde{c}_k - c_k| \leq k \binom{n}{k} \max\{\|A\|_p, \|A+E\|_p\}^{k-1} \|E\|_p$$

Proof: Fréchet derivatives of wedge products

A Perturbed Coefficient

$$\tilde{c}_k = (-1)^k \sum_{1 \le i_1 < \ldots < i_{n-k} \le n} \det(A_{i_1 \ldots i_{n-k}} + E_{i_1 \ldots i_{n-k}})$$

- Sum of $\binom{n}{k}$ minors
- Each minor is determinant of matrix of order k
- Apply determinant bound to each term
- Bounds contain k largest singular values $\sigma_1, \ldots, \sigma_k$

Bound for Perturbed Coefficient

- Elementary symmetric function of k largest singular values $\sigma_1, \ldots, \sigma_k$

$$s_j^{(k)} = \sum_{1 \le i_1 < \ldots < i_j \le k} \sigma_{i_1} \ldots \sigma_{i_j}$$

- Perturbation bound

$$|\tilde{c}_k - c_k| \le \binom{n}{k} \left(s_{k-1}^{(k)} \|E\| + \cdots + s_1^{(k)} \|E\|^{k-1} + \|E\|^k \right)$$

- First order bound

$$|\tilde{c}_k - c_k| \le \binom{n}{k} s_{k-1}^{(k)} \|E\| + \mathcal{O}(\|E\|^2)$$

where $s_{k-1}^{(k)} \le k\, \sigma_1 \ldots \sigma_{k-1}$

Normal (or Hermitian) Matrices

- Perturbation bound

$$|\tilde{c}_k - c_k| \leq (n-k+1)s_{k-1}\,\|E\| + \cdots + \binom{n-1}{k-1}s_1\,\|E\|^{k-1} + \binom{n}{k}\|E\|^k$$

- Elementary functions in terms of all singular values (not just the largest ones)
- Binomial factors smaller than in general bound
- First order bound

$$|\tilde{c}_k - c_k| \leq (n-k+1)s_{k-1}\,\|E\| + \mathcal{O}\left(\|E\|^2\right)$$

- Normal (or Hermitian) matrices may have better conditioned coefficients

Summary

- Characteristic polynomial of $n \times n$ complex matrix A

$$\det(\lambda I - A) = \lambda^n + c_1 \lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n$$

- $c_k \longrightarrow$ elementary symmetric functions of eigenvalues
- Perturbation bounds in terms of elementary symmetric functions of singular values
- Absolute local (first order) condition numbers
- Normal (or Hermitian) matrices may have better conditioned coefficients