The Jordan Form of Complex Tridiagonal Matrices

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Goal

Complex tridiagonal matrix

$$\boldsymbol{T} = \begin{pmatrix} \alpha_1 & \beta_1 & & \\ \gamma_1 & \alpha_2 & \ddots & \\ & \ddots & \ddots & \beta_{n-1} \\ & & \gamma_{n-1} & \alpha_n \end{pmatrix}$$

Jordan decomposition $\boldsymbol{T} = \boldsymbol{X}\boldsymbol{J}\boldsymbol{X}^{-1}$

Express $\boldsymbol{X}$ in terms of α_j, β_j and γ_j

Overview

Idea:

- Eigenvectors $=$ columns of adjoint
- Principal vectors $=$ derivatives of eigenvectors

Talk:

- Adjoints (adjugates)
- Eigenvectors from adjoints
- Principal vectors as derivatives
- Tridiagonal Matrices

Adjoints (Adjugates)

$n \times n$ complex matrix A

$$\mathrm{adj}(A)\, A = A\, \mathrm{adj}(A) = \det(A) I_n$$

- $\mathrm{rank}(A) = n$: $\mathrm{rank}\,(\mathrm{adj}(A)) = n$
- $\mathrm{rank}(A) = n - 1$: $\mathrm{rank}\,(\mathrm{adj}(A)) = 1$
- $\mathrm{rank}(A) \leq n - 2$: $\mathrm{adj}(A) = 0$

Element (i, j) of $\mathrm{adj}(A)$: $(-1)^{i+j} \det(A_{ji})$

Examples

$$A = \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix} \quad \operatorname{adj}(A) = \begin{pmatrix} 2 \cdot 3 & & \\ & 1 \cdot 3 & \\ & & 1 \cdot 2 \end{pmatrix}$$

$$A = \begin{pmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{pmatrix} \qquad \operatorname{adj}(A) = \begin{pmatrix} \lambda^2 & -\lambda & 1 \\ & \lambda^2 & -\lambda \\ & & \lambda^2 \end{pmatrix}$$

Eigenvectors from Adjoints

Complex square matrix A, complex scalar ξ

$$(A - \xi I)\,\mathrm{adj}(A - \xi I) = \det(A - \xi I) I$$

ith column:

$$(A - \xi I)\,\underbrace{\mathrm{adj}(A - \xi I) e_i}_{x(\xi)} = \det(A - \xi I) e_i$$

λ eigenvalue of A:

$$(A - \lambda I)\, x(\lambda) \;=\; \underbrace{\det(A - \lambda I)}_{=0}\, e_i$$

When is $x(\lambda)$ an Eigenvector?

λ eigenvalue of A, $\quad x(\lambda) \equiv \mathrm{adj}(A - \lambda I)e_i$

$$(A - \lambda I)\, x(\lambda) = 0$$

$x(\lambda)$ eigenvector

- if $\quad x(\lambda) \neq 0$
- if $\quad \mathrm{adj}(A - \lambda I) \neq 0$
- if $\quad \mathrm{rank}\,(\mathrm{adj}(A - \lambda I)) = 1$
- if $\quad \lambda$ has geometric multiplicity 1

Eigenvectors from Adjoints

Given: complex matrix A with eigenvalue λ

If λ has geometric multiplicity 1 then

- $\mathrm{adj}(A - \lambda I) \neq 0$
- Non-zero columns of $\mathrm{adj}(A - \lambda I)$ are eigenvectors of A
- $\underbrace{\mathrm{adj}(A - \lambda I) e_i}_{x(\lambda)} \neq 0$ for some i

$$(A - \lambda I) x(\lambda) = 0$$

[Gantmacher 1959]

Principal Vectors from Adjoints

$$(A - \xi I) \underbrace{\operatorname{adj}(A - \xi I) e_i}_{x(\xi)} = \det(A - \xi I) e_i$$

Differentiate

$$-x(\xi) + (A - \xi I)\, x'(\xi) = \frac{d}{d\xi} \det(A - \xi I) e_i$$

λ multiple eigenvalue:

$$-x(\lambda) + (A - \lambda I) x'(\lambda) = \underbrace{\frac{d}{d\xi} \det(A - \xi I)|_{\xi=\lambda}}_{=0} e_i$$

When is $x'(\lambda)$ Principal Vector?

Given: complex matrix with eigenvalue $\boldsymbol{\lambda}$

If $\boldsymbol{\lambda}$ has geometric multiplicity $\mathbf{1}$
and algebraic multiplicity $\geq \mathbf{2}$ then

- $\boldsymbol{x(\lambda) \equiv \mathrm{adj}(A - \lambda I) e_i \neq 0}$ for some $\boldsymbol{i}$
- $\boldsymbol{x'(\lambda) \equiv \frac{d}{d\xi} x(\xi)|_{\xi=\lambda}}$
- $\boldsymbol{(A - \lambda I)\, x'(\lambda) = x(\lambda)}$

Principal vectors are derivatives of
columns of $\mathbf{adj}(\boldsymbol{A - \lambda I})$

Higher Order Derivatives

$$x_1(\xi) \equiv \operatorname{adj}(A - \xi I)e_i$$

$$(A - \xi I)x_1(\xi) = \det(A - \xi I)e_i$$

Differentiate j times

$$(A - \xi I)x_{j+1}(\xi) = x_j(\xi) - \frac{1}{j!}\frac{d^j}{d\xi^j}\det(A - \xi I)e_i$$

where $\quad x_{j+1}(\xi) \equiv \frac{1}{j}\, x_j'(\xi)$

Higher Order Principal Vectors

Given: complex matrix A with eigenvalue λ

If λ has geometric multiplicity 1
and algebraic multiplicity $k \geq 2$ then

- $x_1(\xi) \equiv \mathrm{adj}(A - \xi I)e_i$
- $x_{j+1}(\lambda) \equiv \frac{1}{j} x_j'(\lambda)$, $1 \leq j \leq k-1$
- $(A - \lambda I)\, x_{j+1}(\lambda) \; = x_j(\lambda)$

Principal vectors of order j are jth derivatives of columns of $\mathrm{adj}(A - \lambda I)$

Jordan Block

$$A = \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix} \quad \mathrm{adj}(A - \xi I) = \begin{pmatrix} \xi^2 & \xi & 1 \\ 0 & \xi^2 & \xi \\ 0 & 0 & \xi^2 \end{pmatrix}$$

$$x_1(\xi) \equiv \begin{pmatrix} 1 \\ \xi \\ \xi^2 \end{pmatrix} \qquad x_2(\xi) = x_1'(\xi) = \begin{pmatrix} 0 \\ 1 \\ 2\xi \end{pmatrix}$$

$$x_3(\xi) = \frac{1}{2}\, x_2'(\xi) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Adjoints & Eigen/Principal Vectors

Given: complex matrix A

$$(A - \xi I)\ \mathrm{adj}(A - \xi I) = \det(A - \xi I) I$$

If eigenvalue λ of A has geometric multiplicity 1

- Eigenvectors are non-zero columns of $\mathrm{adj}(A - \lambda I)$
- Principal vectors are derivatives of columns of $\mathrm{adj}(A - \lambda I)$

Complex Tridiagonal Matrices

$$T = \begin{pmatrix} \alpha_1 & \beta_1 & & \\ \gamma_1 & \alpha_2 & \ddots & \\ & \ddots & \ddots & \beta_{n-1} \\ & & \gamma_{n-1} & \alpha_n \end{pmatrix}$$

Unreduced: $\beta_j \neq 0$, $\gamma_j \neq 0$

All eigenvalues have geometric multiplicity 1

Characteristic Polynomials

Leading principal submatrix of order j

$$T_j = \begin{pmatrix} \alpha_1 & \beta_1 & & \\ \gamma_1 & \alpha_2 & \ddots & \\ & \ddots & \ddots & \beta_{j-1} \\ & & \gamma_{j-1} & \alpha_j \end{pmatrix}$$

Characteristic polynomial $\phi_j(\xi) \equiv \det(T_j - \xi I)$

Recurrence:

$$\phi_j(\xi) = (\alpha_j - \xi)\, \phi_{j-1}(\xi) - \beta_{j-1}\, \gamma_{j-1}\, \phi_{j-2}(\xi)$$

Adjoints and Tridiagonals

$n \times n$ complex tridiagonal T

$$(T - \xi I)\, x(\xi) = \phi_n(\xi)\, e_n$$

where $\quad \phi_n(\xi) = \det(T - \xi I)$

$$x(\xi) = \operatorname{adj}(T - \xi I) e_n = \begin{pmatrix} \beta_1 \cdots \beta_{n-1} \\ \beta_2 \cdots \beta_{n-1}\, \phi_1(\xi) \\ \vdots \\ \beta_{n-1}\, \phi_{n-2}(\xi) \\ \phi_{n-1}(\xi) \end{pmatrix}$$

Eigenvector

$$(T - \lambda I)\, x(\lambda) = 0$$

$$x(\lambda) \;=\; \begin{pmatrix} \beta_1 \cdots \beta_{n-1} \\ \beta_2 \cdots \beta_{n-1}\, \phi_1(\lambda) \\ \vdots \\ \beta_{n-1}\, \phi_{n-2}(\lambda) \\ \phi_{n-1}(\lambda) \end{pmatrix}$$

Super-diagonal elements $\beta_1, \ldots, \beta_{n-1}$

[Wilkinson 1965: Hermitian tridiagonals]

Principal Vectors

$n \times n$ unreduced complex tridiagonal T
Super-diagonal elements $\beta_1, \ldots, \beta_{n-1}$
Eigenvalue λ of multiplicity > 1

- Eigenvector $x_1(\lambda)$: $\quad (T - \lambda I)\, x_1(\lambda) = 0$
- First principal vector $x_2(\lambda)$:

$$(T - \lambda I)\, x_2(\lambda) = x_1(\lambda)$$

- Second principal vector $x_3(\lambda)$:

$$(T - \lambda I)\, x_3(\lambda) = x_2(\lambda)$$

Eigenvector

$$(T - \lambda I)\, x_1(\lambda) = 0$$

$$x_1(\lambda) \;=\; \begin{pmatrix} \beta_1 \cdots \beta_{n-1} \\ \beta_2 \cdots \beta_{n-1}\, \phi_1(\lambda) \\ \vdots \\ \beta_{n-1}\, \phi_{n-2}(\lambda) \\ \phi_{n-1}(\lambda) \end{pmatrix}$$

Determinants $\phi_1(\lambda), \ldots, \phi_{n-1}(\lambda)$

First Principal Vector

$$(T - \lambda I)\, x_2(\lambda) = x_1(\lambda)$$

$$x_2(\lambda) = \begin{pmatrix} 0 \\ \beta_2 \cdots \beta_{n-1} \\ \beta_3 \cdots \beta_{n-1}\, \phi_2'(\lambda) \\ \vdots \\ \beta_{n-1}\, \phi_{n-2}'(\lambda) \\ \phi_{n-1}'(\lambda) \end{pmatrix}$$

First derivatives $\phi_2'(\lambda), \ldots, \phi_{n-1}'(\lambda)$

Second Principal Vector

$$(T - \lambda I)\, x_3(\lambda) = x_2(\lambda)$$

$$x_3(\lambda) = \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ \beta_3 \cdots \beta_{n-1} \\ \beta_4 \cdots \beta_{n-1}\, \phi_3''(\lambda) \\ \vdots \\ \beta_{n-1}\, \phi_{n-2}''(\lambda) \\ \phi_{n-1}''(\lambda) \end{pmatrix}$$

Second derivatives $\phi_3''(\lambda), \ldots, \phi_{n-1}''(\lambda)$

jth Principal Vector

$$(T - \lambda I)\, x_{j+1}(\lambda) = x_j(\lambda)$$

$$x_{j+1}(\lambda) = \frac{1}{j!} \begin{pmatrix} 0_j \\ \beta_{j+1} \cdots \beta_{n-1} \\ \beta_{j+2} \cdots \beta_{n-1}\, \phi_{j+1}^{(j)}(\lambda) \\ \vdots \\ \beta_{n-1}\, \phi_{n-2}^{(j)}(\lambda) \\ \phi_{n-1}^{(j)}(\lambda) \end{pmatrix}$$

jth derivatives $\quad \phi_{j+1}^{(j)}(\lambda), \ldots, \phi_{n-1}^{(j)}(\lambda)$

All Principal Vectors

Unreduced complex tridiagonal T

Eigenvalue λ of multiplicity $k > 1$

$$T\,X \;=\; X\,J_k(\lambda) \qquad X = \begin{pmatrix} * & & \\ \vdots & \ddots & \\ * & \cdots & * \\ * & \cdots & * \\ \vdots & \vdots & \vdots \\ * & \cdots & * \end{pmatrix}$$

Complex Symmetric Tridiagonals

$$\boldsymbol{T}\,\boldsymbol{X} \;=\; \boldsymbol{X}\,\boldsymbol{J}_k(\boldsymbol{\lambda})$$

$$\boldsymbol{X} = \begin{pmatrix} * & & \\ \vdots & \ddots & \\ * & \cdots & * \\ * & \cdots & * \\ \vdots & \vdots & \vdots \\ * & \cdots & * \end{pmatrix} \qquad \boldsymbol{X}^T\boldsymbol{X} \;=\; \begin{pmatrix} & & * \\ & \iddots & \vdots \\ * & \cdots & * \end{pmatrix}$$

Certain principal vectors are orthogonal

Complex Symmetric Tridiagonals

Eigenvector $\boldsymbol{x(\lambda)}$: $\quad (\boldsymbol{T} - \boldsymbol{\lambda I})\boldsymbol{x}(\boldsymbol{\lambda}) = \mathbf{0}$
Characteristic polynomial: $\phi_n(\xi) = \det(\boldsymbol{T} - \xi \boldsymbol{I})$

$$\boldsymbol{x}(\boldsymbol{\lambda})^T \boldsymbol{x}(\boldsymbol{\lambda}) = \phi_n'(\boldsymbol{\lambda})\phi_{n-1}(\boldsymbol{\lambda})$$

$\boldsymbol{\lambda}$ has algebraic multiplicity > 1: $\phi_n'(\boldsymbol{\lambda}) = \mathbf{0}$

$$\boldsymbol{x}(\boldsymbol{\lambda})^T \boldsymbol{x}(\boldsymbol{\lambda}) = \mathbf{0}$$

Eigenvector $\boldsymbol{x}(\boldsymbol{\lambda})$ isotropic (asymptotic)

[Glidman 1965, Craven 1969, Scott 1993]

Example

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \imath \\ 0 & \imath & 0 \end{pmatrix} \qquad \imath = \sqrt{-1}$$

$$T = XJ_3(0)X^{-1}$$

$$X = \begin{pmatrix} \imath & 0 & 0 \\ 0 & \imath & 0 \\ -1 & 0 & 1 \end{pmatrix} \qquad X^T X = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

Summary

- Complex square matrices A
- Eigenvalues λ of geometric multiplicity 1
- Adjoint of $A - \lambda I$ non-zero
- Eigenvectors: columns of adjoint
- Principal vectors: derivatives of eigenvectors
- Analogous: left eigen/principal vectors

Summary, ctd

- Unreduced complex tridiagonals
- All eigenvalues have geometric multiplicity 1
- Explicit expressions for eigenvectors and principal vectors
- Symmetric tridiagonals with eigenvalues of multiplicity > 1
- Eigenvectors isotropic
- Certain principal vectors are orthogonal