Lower Bounds for the Smallest Eigenvalue of a Symmetric Matrix

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Overview

- Numerical continuation
- Singularity: simple fold
- Bounds for Jacobians
- Idea: SVD of Jacobian
- Eigenvalue Bounds

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- Bounds for Jacobians
- Idea: SVD of Jacobian
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Assumptions:

- Exact arithmetic
- Jacobians computed exactly

Numerical Continuation

Solve system of nonlinear equations

$$\boldsymbol{G(u, \lambda) = 0}$$

for various values of real parameter $\boldsymbol{\lambda}$

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- Parameter continuation
- Singularity: simple fold
- Pseudo-arclength continuation

Parameter Continuation

Given:

Function $\boldsymbol{G} : \mathbb{R}^{\boldsymbol{n+1}} \longrightarrow \mathbb{R}^{\boldsymbol{n}}$

Want:

$\boldsymbol{u} \in \mathbb{R}^{\boldsymbol{n}}$ such that $\boldsymbol{G}(\boldsymbol{u}, \boldsymbol{\lambda}) = \mathbf{0}$
for $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{\mathbf{0}} +$ multiples of $\boldsymbol{d\lambda}$

Parameter Continuation

Given:

Function $G : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$

Want:

$u \in \mathbb{R}^n$ such that $G(u, \lambda) = 0$
for $\lambda = \lambda_0 +$ multiples of $d\lambda$

$\lambda = \lambda_0$
Repeat:

Solve $G(u, \lambda) = 0$ for u
with initial iterate u_0

$u_0 = u$
$\lambda = \lambda + d\lambda$

Solve $G(u, \lambda) = 0$ for u

Jacobian with respect to u: $G_u \equiv \frac{\partial G}{\partial u}$

Newton's method:

$$u_{k+1} = u_k - G_u(u_k, \lambda)^{-1} G(u_k, \lambda), \qquad k \geq 0$$

Solve $G(u, \lambda) = 0$ for u

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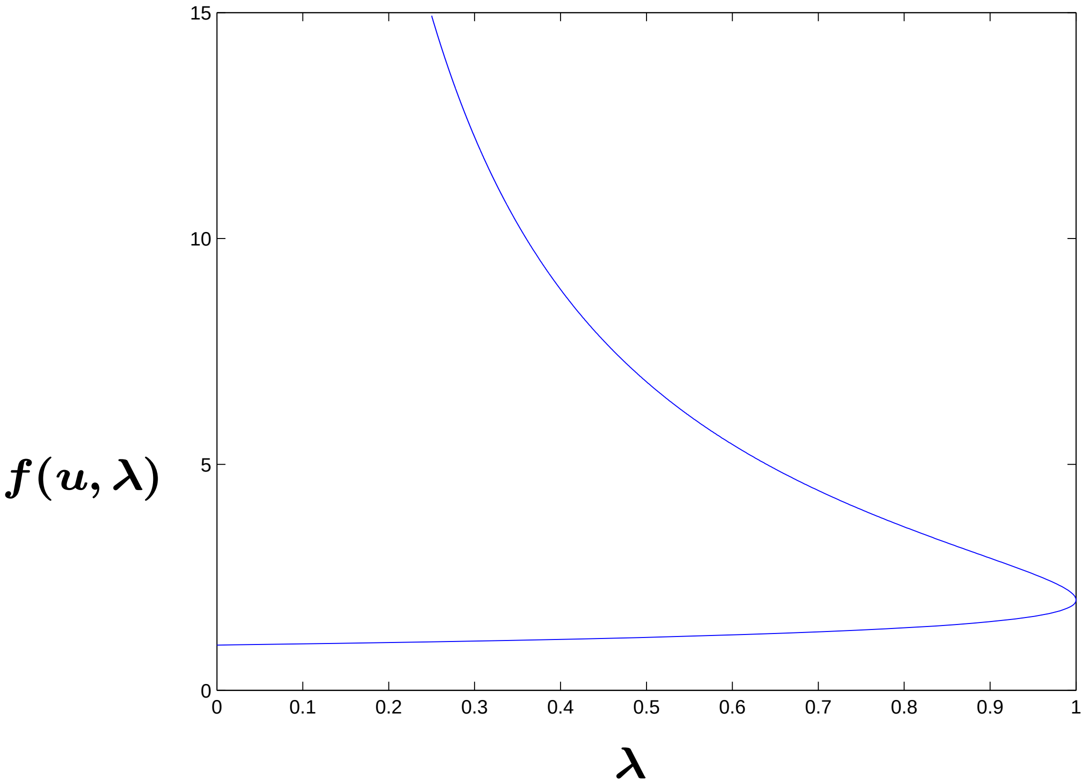
Newton's method:

$$u_{k+1} = u_k - G_u(u_k, \lambda)^{-1}\, G(u_k, \lambda), \qquad k \geq 0$$

Quadratic convergence requires:

- $G(u_0, \lambda_0) = 0$
- $G_u(u_0, \lambda_0)$ nonsingular
- $|\lambda - \lambda_0| \leq c / \, \|G_u^{-1}(u_0, \lambda_0)\|$

Example: Singular Jacobian



Jacobian G_u singular at $\lambda = 1$ (simple fold)

Simple Fold

$G(u, \lambda) = 0$ has simple fold at solution (u_0, λ_0) if

1. Null space of G_u has dimension 1
2. G_λ not in range of G_u

How to avoid a simple fold?

Simple Fold

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How to avoid a simple fold?

- Matrix $\begin{pmatrix} G_u & G_\lambda \end{pmatrix}$ has full row rank
- Add row to get a nonsingular Jacobian
- Solve instead for u and λ
- Introduce new parameter

Arclength

- New parameter: arclength s
- Differentiate $G(u_0, \lambda_0) = 0$ with respect to s

$$G_u \dot{u}_0 + G_\lambda \dot{\lambda}_0 = 0$$

- Nonsingular Jacobian $\begin{pmatrix} G_u & G_\lambda \\ \dot{u}_0^T & \dot{\lambda}_0 \end{pmatrix}$
- Pseudo-arclength normalization

$$g(u, \lambda, s) \equiv \dot{u}_0^T (u - u_0) + \dot{\lambda}_0 (\lambda - \lambda_0) + (s - s_0)$$

Extended System

$$x \equiv \begin{pmatrix} u \\ \lambda \end{pmatrix} \qquad F(x, s) = \begin{pmatrix} G(x) \\ g(x, s) \end{pmatrix}$$

- Old problem: solve $G(u, \lambda) = 0$
- New problem: solve $F(x, s) = 0$

Pseudo-arclength continuation:

Solve $F(x, s) = 0$ for $s = s_0 +$ multiples of ds

by parameter continuation

Pseudo-Arclength Continuation

Jacobian with respect to x: $\boldsymbol{F_x \equiv \frac{\partial F}{\partial x}}$

Newton's method:

$$\boldsymbol{x_{k+1} = x_k - F_x(x_k, s)^{-1}\, F(x_k, s), \qquad k \geq 1}$$

Pseudo-Arclength Continuation

Jacobian with respect to x: $F_x \equiv \frac{\partial F}{\partial x}$

Newton's method:

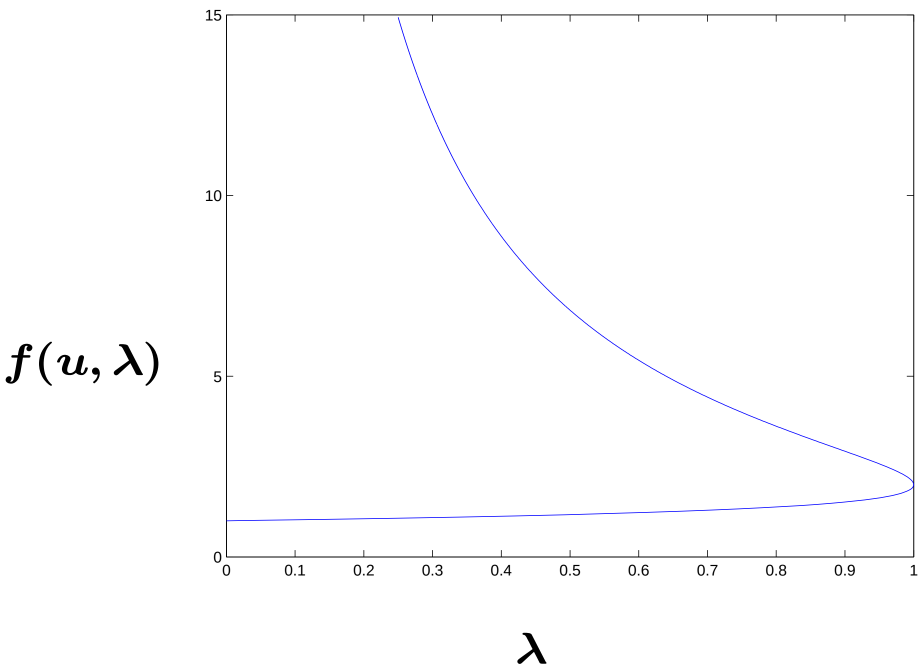
$$x_{k+1} = x_k - F_x(x_k, s)^{-1} F(x_k, s), \qquad k \geq 1$$

Quadratic convergence requires:

- $F(x_0, s_0) = 0$
- $G_u(x_0)$ nonsingular, or x_0 simple fold
- $|s - s_0| \leq c / \|F_x^{-1}(x_0, s_0)\|$

Summary

- Original system: $G(u, \lambda) = 0$
- Problem: G_u singular at simple fold



Summary, ctd

- New parameter: arclength s
- Extended system: $F(x,s)=0$

$$x=\begin{pmatrix}u\\ \lambda\end{pmatrix}\qquad F(x,s)=\begin{pmatrix}G(x)\\ g(x,s)\end{pmatrix}$$

- Newton's method requires:
  $G_u(x_0)$ nonsingular, or x_0 simple fold

$$s-s_0\le c/\,\|F_x^{-1}(x_0,s_0)\|$$

To control step length need bounds for $\|F_x^{-1}\|$

Upper Bound on $\|F_x^{-1}\|$

$$F = \begin{pmatrix} G(u, \lambda) \\ g(u, \lambda, s) \end{pmatrix} \qquad F_x = \begin{pmatrix} G_u & G_\lambda \\ g_u & g_\lambda \end{pmatrix}$$

$$F_x F_x^T = \begin{pmatrix} G_u G_u^T + G_\lambda G_\lambda^T & 0 \\ 0 & 1 \end{pmatrix}$$

- G_u has null space of dimension ≤ 1
- Bound $\|(G_u G_u^T + G_\lambda G_\lambda^T)^{-1}\|$
- Use SVD of G_u

SVD of G_u

G_u nonsingular, or null space of dimension 1

- Singular values:
  $\sigma_1 \geq \cdots \geq \sigma_{n-1} > 0, \quad \sigma_n \geq 0$
- v is left singular vector for σ_n:
  If $\sigma_n = 0$ then $v^T G_u = 0$
- G_λ not in range of G_u: $\quad v^T G_\lambda \neq 0$

SVD of G_u

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$G(u, \lambda) = 0$ has simple fold if

1. $\sigma_{n-1} > \sigma_n = 0$
2. $v^T G_\lambda \neq 0$

SVD $\longrightarrow$ Eigenvalue Problem

Want: upper bound for

$$\|(G_u G_u^T + G_\lambda G_\lambda^T)^{-1}\| = 1/\lambda_{\min}\left(G_u G_u^T + G_\lambda G_\lambda^T\right)$$

- $G_u G_u^T$ symmetric positive semi-definite
- Eigenvalues $\sigma_1^2 \geq \ldots \geq \sigma_{n-1}^2 > 0, \quad \sigma_n^2 \geq 0$
- $G_u G_u^T v = \sigma_n^2 v$
- $\sigma_n > 0 \quad$ or $\quad v^T G_\lambda \neq 0$

Need: Eigenvalues of rank-one update

Rank-One Updates

Real symmetric matrix $\boldsymbol{A}$
Eigenvalues $\boldsymbol{\alpha_1 \geq \ldots \geq \alpha_n}$
Real vector $\boldsymbol{y}$

Weyl's Theorem:

$$\boldsymbol{\alpha_n \leq \lambda_{\min}(A + yy^T) \leq \alpha_{n-1}}$$

Want: Lower bound as a function of $\boldsymbol{y}$

Eigenvalue Bound

- Eigenvalues: $\alpha_{n-1} > 0 \quad \alpha_n \geq 0$
  $$\text{gap} \equiv \alpha_{n-1} - \alpha_n$$
- Eigenvector: $Av = \alpha_n v \quad \|v\|_2 = 1$
- Relation between y and v:

$$y_n \equiv |y^T v| \quad \xi \equiv y_n + \sqrt{\|y\|_2^2 - y_n^2}$$

- Simple fold: $\alpha_n > 0$ or $y_n \neq 0$

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$$\lambda_{\min}(A + yy^T) \geq \min\left\{\alpha_n + y_n^2 \frac{\text{gap}}{\text{gap}+\xi^2},\ \alpha_{n-1}\frac{y_n^2}{\xi^2}\right\}$$

Simpler Eigenvalue Bound

- Lower bound for $\lambda_{\min}(A + yy^T)$
$$\min\left\{\alpha_n + y_n^2 \frac{\text{gap}}{\text{gap}+\xi^2},\ y_n^2\ \frac{\alpha_{n-1}}{\xi^2}\right\}$$
- Non-negative eigenvalues
$$\frac{\alpha_{n-1}}{\xi^2} \geq \frac{\text{gap}}{\text{gap}+\xi^2}$$
- Weaker lower bound
$$\lambda_{\min}(A + yy^T) \geq y_n^2 \frac{\text{gap}}{\text{gap}+\xi^2}$$
- Weyl's Theorem:
$$\min(A + yy^T) \geq \max\left\{\alpha_n,\ y_n^2 \frac{\text{gap}}{\text{gap}+\xi^2}\right\}$$

Bound for Jacobians

- Singular values of G_u: $\quad \sigma_{n-1} > 0 \quad \sigma_n \geq 0$
  $$\text{gap} \equiv \sigma_{n-1}^2 - \sigma_n^2$$
- v is left singular vector of σ_n
- Relation between G_λ and v:
  $$\xi \equiv |v^T G_\lambda| + \|(I - vv^T)\, G_\lambda\|$$
- Simple fold: $\sigma_n > 0$ or $v^T G_\lambda \neq 0$

Bound for Jacobians

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- Simple fold: $\sigma_n > 0$ or $v^T G_\lambda \neq 0$

$$\|F_x^{-1}\| \leq \max\{1, \tfrac{1}{\sqrt{\alpha}}\}$$

where $\alpha = \max\left\{\sigma_n^2,\ |v^T G_\lambda|^2 \frac{\text{gap}}{\text{gap}+\xi^2}\right\}$

Summary: Problem

- Numerical continuation $G(u, \lambda) = 0$
- Jacobian G_u singular at simple fold
- Pseudo-arclength continuation

$$F(x, s) = 0 \qquad x = \begin{pmatrix} u \\ \lambda \end{pmatrix}$$

- Jacobian F_x nonsingular
- Need to control step length for s
- Need bounds for $||F_x^{-1}||$

Summary: Contributions

- New characterization of simple fold

  In terms of SVD of G_u
- Bound for $\|F_x^{-1}\|$

  Singular values of G_u

  Relation of G_λ to null space of G_u
- New lower bounds for $\lambda_{\min}(A + yy^T)$

  Gap: $\lambda_{\min}(A)$ and next eigenvalue

  Relation of y to eigen space of $\lambda_{\min}(A)$