# The Mathematics Behind Google’s PageRank

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# Two Factors

Determine where Google displays a web page on the *Search Engine Results Page*:

1. PageRank (links)

   A page has high PageRank
   if many pages with high PageRank link to it

2. Hypertext Analysis (page contents)

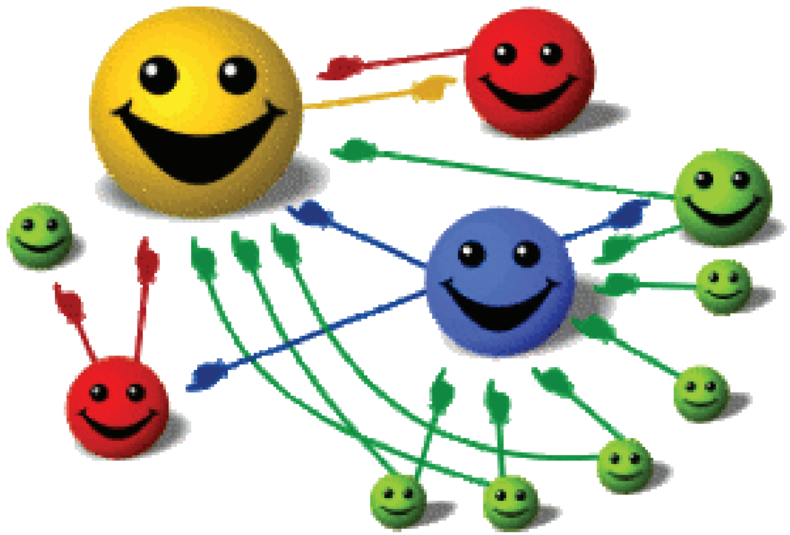
   Text, fonts, subdivisions, location of words, contents of neighbouring pages

# PageRank

*An objective measure of the citation importance of a web page* [Brin & Page 1998]

- Assigns a rank to every web page
- Influences the order in which Google displays search results
- Based on link structure of the web graph
- Does not depend on contents of web pages
- Does not depend on query

# More PageRank More Visitors



# PageRank

. . . *continues to provide the basis for all of our web search tools* http://www.google.com/technology/

- “Links are the currency of the web”
- Exchanging & buying of links
- BO (backlink obsession)
- Search engine optimization

# Overview

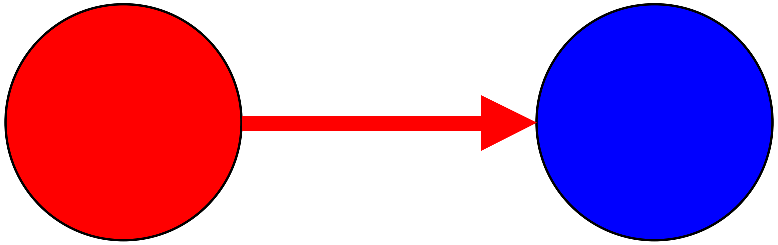
- Mathematical Model of Internet
- Computation of PageRank
- Is the Ranking Correct?
- Floating Point Arithmetic Issues

# Mathematical Model of Internet

1. Represent internet as graph
2. Represent graph as stochastic matrix
3. Make stochastic matrix more convenient
   $\Longrightarrow$ Google matrix
4. dominant eigenvector of Google matrix
   $\Longrightarrow$ PageRank

# The Internet as a Graph

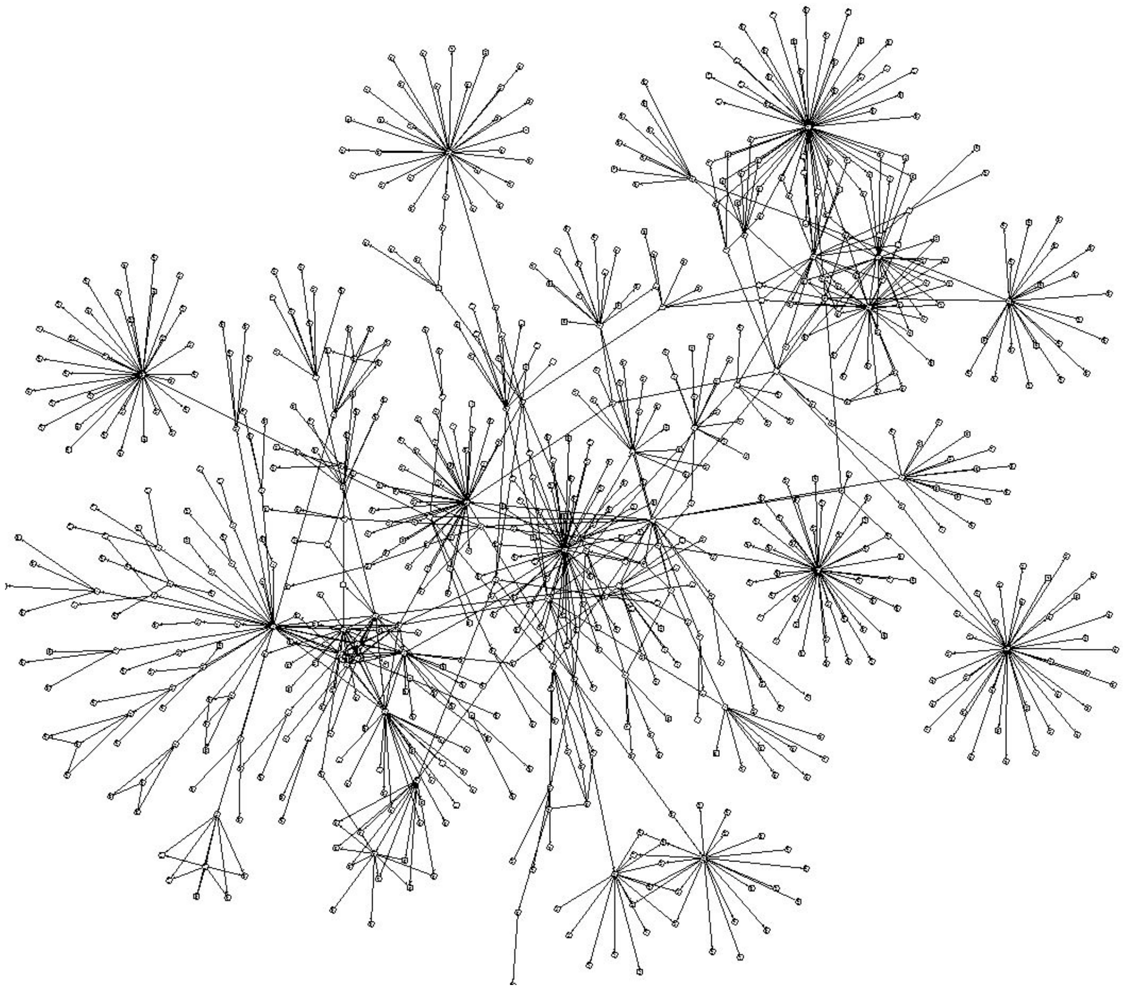
Link from one web page to another web page



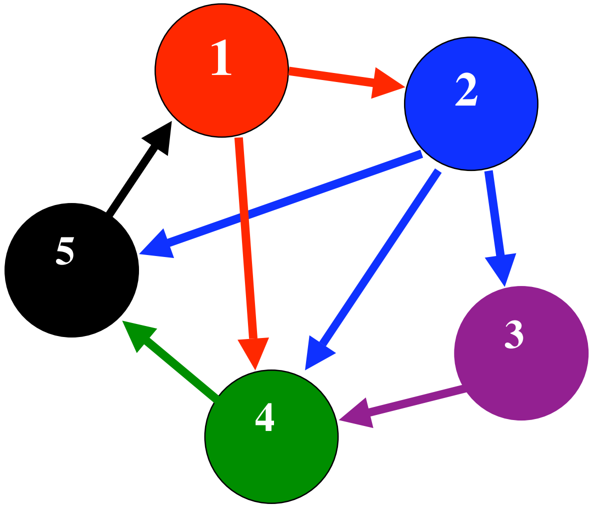
Web graph:

Web pages = nodes
Links = edges

# The Internet as a Graph



# The Web Graph as a Matrix




$$S = \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Links = nonzero elements in matrix

# Properties of Matrix $S$

- Row $i$ of $S$: Links from page $i$ to other pages
- Column $i$ of $S$: Links into page $i$
- $S$ is a stochastic matrix:
  All elements in $[0, 1]$
  Elements in each row sum to 1
- Dominant left eigenvector:

$$\omega^T S = \omega^T \qquad \omega \geq 0 \qquad \|\omega\|_1 = 1$$

- $\omega_i$ is probability of visiting page $i$
- But: $\omega$ not unique

# Google Matrix

Convex combination

$$G = \alpha S + \underbrace{(1 - \alpha)\mathbb{1} v^T}_{\text{rank } 1}$$

- Stochastic matrix $S$
- Damping factor $0 \leq \alpha < 1$
  e.g. $\alpha = .85$
- Column vector of all ones $\mathbb{1}$
- Personalization vector $v \geq 0 \quad \|v\|_1 = 1$
  Models teleportation

# PageRank

$$G = \alpha S + (1-\alpha)\mathbb{1}v^T$$

- $G$ is stochastic, with eigenvalues:

$$1 > \alpha|\lambda_2(S)| \geq \alpha|\lambda_3(S)| \geq \dots$$

- Unique dominant left eigenvector:

$$\pi^T G = \pi^T \qquad \pi \geq 0 \qquad \|\pi\|_1 = 1$$

- $\pi_i$ is PageRank of web page $i$

[Haveliwala & Kamvar 2003, Eldén 2003,

Serra-Capizzano 2005]

# How Google Ranks Web Pages

- Model:
  Internet $\longrightarrow$ web graph $\longrightarrow$ stochastic matrix $G$
- Computation:
  PageRank $\pi$ is eigenvector of $G$

  $\pi_i$ is PageRank of page $i$
- Display:

  If $\pi_i > \pi_k$ then
  page $i$ may* be displayed before page $k$

* depending on hypertext analysis

# Facts

- The anatomy of a large-scale hypertextual web search engine [Brin & Page 1998]
- US patent for PageRank granted in 2001
- Google indexes 10's of billions of web pages (1 billion = $10^9$)
- Google serves $\geq 200$ million queries per day
- Each query processed by $\geq 1000$ machines
- All search engines combined process more than $500$ million queries per day

[Desikan, 26 October 2006]

# Computation of PageRank

*The world's largest matrix computation*
[Moler 2002]

- Eigenvector
- Matrix dimension is 10's of billions
- The matrix changes often
  250,000 new domain names every day
- Fortunately: Matrix is sparse

# Power Method

Want: $\pi$ such that $\quad \pi^T G = \pi^T$

Power method:

Pick an initial guess $x^{(0)}$
Repeat

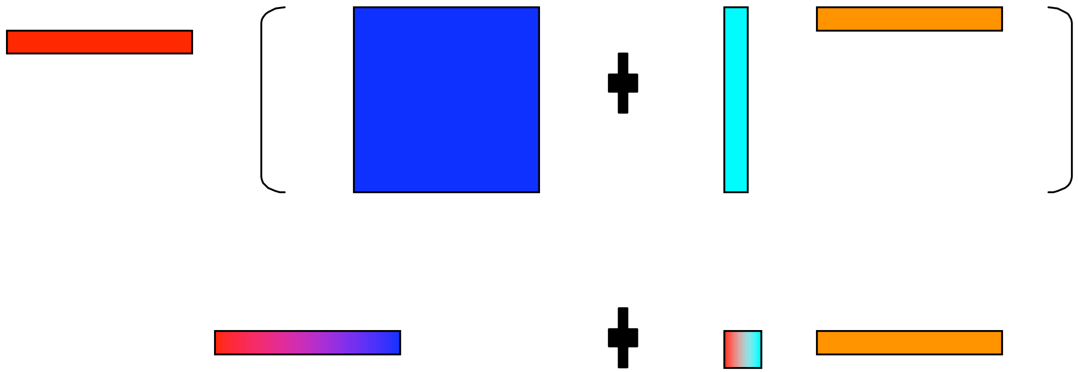
$$[x^{(k+1)}]^T := [x^{(k)}]^T G$$

until "termination criterion satisfied"

Each iteration is a matrix vector multiply

# Matrix Vector Multiply

$$x^T G = \quad x^T \left[\alpha S + (1-\alpha)\mathbb{1} v^T\right]$$



Cost: # non-zero elements in $S$

A power method iteration is cheap

# Error Reduction in 1 Iteration

$$\pi^T G = \pi^T \qquad G = \alpha S + (1-\alpha)\mathbb{1} v^T$$

$$\begin{aligned} [x^{(k+1)} - \pi]^T &= [x^{(k)}]^T G - \pi^T G \\ &= \alpha [x^{(k)} - \pi]^T S \end{aligned}$$

Error: $\underbrace{\|x^{(k+1)} - \pi\|_1}_{\text{iteration } k+1} \leq \alpha \underbrace{\|x^{(k)} - \pi\|_1}_{\text{iteration } k}$

# Error in Power Method

$$\pi^T G = \pi^T \qquad G = \alpha S + (1-\alpha)\mathbb{1} v^T$$

Error after $k$ iterations:

$$\|x^{(k)} - \pi\|_1 \leq \alpha^k \underbrace{\|x^{(0)} - \pi\|_1}_{\leq 2}$$

[Bianchini, Gori & Scarselli 2003]

Error bound does not depend on matrix dimension

# Advantages of Power Method

- Simple implementation (few decisions)
- Cheap iterations (sparse matvec)
- Minimal storage (a few vectors)
- Robust convergence behaviour
- Convergence rate independent of matrix dimension
- Numerically reliable and accurate
(no subtractions, no overflow)

But: can be slow

# PageRank Computation

- Power method

  Page, Brin, Motwani & Winograd 1999

  Bianchini, Gori & Scarselli 2003

- Acceleration of power method

  Kamvar, Haveliwala, Manning & Golub 2003

  Haveliwala, Kamvar, Klein, Manning & Golub 2003

  Brezinski & Redivo-Zaglia 2004, 2006

  Brezinski, Redivo-Zaglia & Serra-Capizzano 2005

- Aggregation/Disaggregation

  Langville & Meyer 2002, 2003, 2006

  Ipsen & Kirkland 2006

# PageRank Computation

- Methods that adapt to web graph

  Broder, Lempel, Maghoul & Pedersen 2004 Kamvar, Haveliwala & Golub 2004

  Haveliwala, Kamvar, Manning & Golub 2003

  Lee, Golub & Zenios 2003

  Lu, Zhang, Xi, Chen, Liu, Lyu & Ma 2004

  Ipsen & Selee 2006

- Krylov methods

  Golub & Greif 2004

  Del Corso, Gullí, Romani 2006

# PageRank Computation

- Schwarz & asynchronous methods

  Bru, Pedroche & Szyld 2005

  Kollias, Gallopoulos & Szyld 2006

- Linear system solution

  Arasu, Novak, Tomkins & Tomlin 2002

  Arasu, Novak & Tomkins 2003

  Bianchini, Gori & Scarselli 2003

  Gleich, Zukov & Berkin 2004

  Del Corso, Gullí & Romani 2004

  Langville & Meyer 2006

# PageRank Computation

- Surveys of numerical methods:
  Langville & Meyer 2004
  Berkhin 2005
  Langville & Meyer 2006 (book)

# Is the Ranking Correct?

$$\pi^T = \begin{pmatrix} .23 & .24 & .26 & .27 \end{pmatrix}$$

- $x^T = \begin{pmatrix} .27 & .26 & .24 & .23 \end{pmatrix}$

  $\|x - \pi\|_\infty = .04$

  Small error, but incorrect ranking

- $y^T = \begin{pmatrix} 0 & .001 & .002 & .997 \end{pmatrix}$

  $\|y - \pi\|_\infty = .727$

  Large error, but correct ranking

# What is Important?

Numerical value $\leftrightarrow$ ordinal rank

ordinal rank:
position of an element in an ordered list

Very little research on ordinal ranking
Rank-stability, rank-similarity

[Lempel & Moran, 2005]
[Borodin, Roberts, Rosenthal & Tsaparas 2005]

# Ordinal Ranking

Largest element gets $\mathbf{Orank}\ \mathbf{1}$

$$\boldsymbol{\pi}^T = \begin{pmatrix} .23 & .24 & .26 & .27 \end{pmatrix}$$

$$\mathbf{Orank}(\boldsymbol{\pi}_4) = \mathbf{1}, \mathbf{Orank}(\boldsymbol{\pi}_1) = \mathbf{4}$$

- $\boldsymbol{x}^T = \begin{pmatrix} .27 & .26 & .24 & .23 \end{pmatrix}$

  $\mathbf{Orank}(\boldsymbol{x}_1) = \mathbf{1} \neq \mathbf{Orank}(\boldsymbol{\pi}_1) = \mathbf{4}$

- $\boldsymbol{y}^T = \begin{pmatrix} 0 & .001 & .002 & .997 \end{pmatrix}$

  $\mathbf{Orank}(\boldsymbol{y}_4) = \mathbf{1} = \mathbf{Orank}(\boldsymbol{\pi}_4)$

# Problems with Ordinal Ranking

When done with power method:

- Popular termination criteria do not guarantee correct ranking
- Additional iterations can destroy ranking
- Rank convergence depends on: $\alpha$, $v$, initial guess, matrix dimension, structure of web graph
- Even if successive iterates have the same ranking, their ranking may not be correct

[Wills & Ipsen 2007]

# Ordinal Ranking Criterion

Given:

Approximation $\boldsymbol{x}$, $\boldsymbol{x} \geq \mathbf{0}$
Error bound $\boldsymbol{\beta} \geq \|\boldsymbol{x} - \boldsymbol{\pi}\|_1$

Criterion: $\boldsymbol{x}_i > \boldsymbol{x}_j + \boldsymbol{\beta} \Longrightarrow \boldsymbol{\pi}_i > \boldsymbol{\pi}_j$

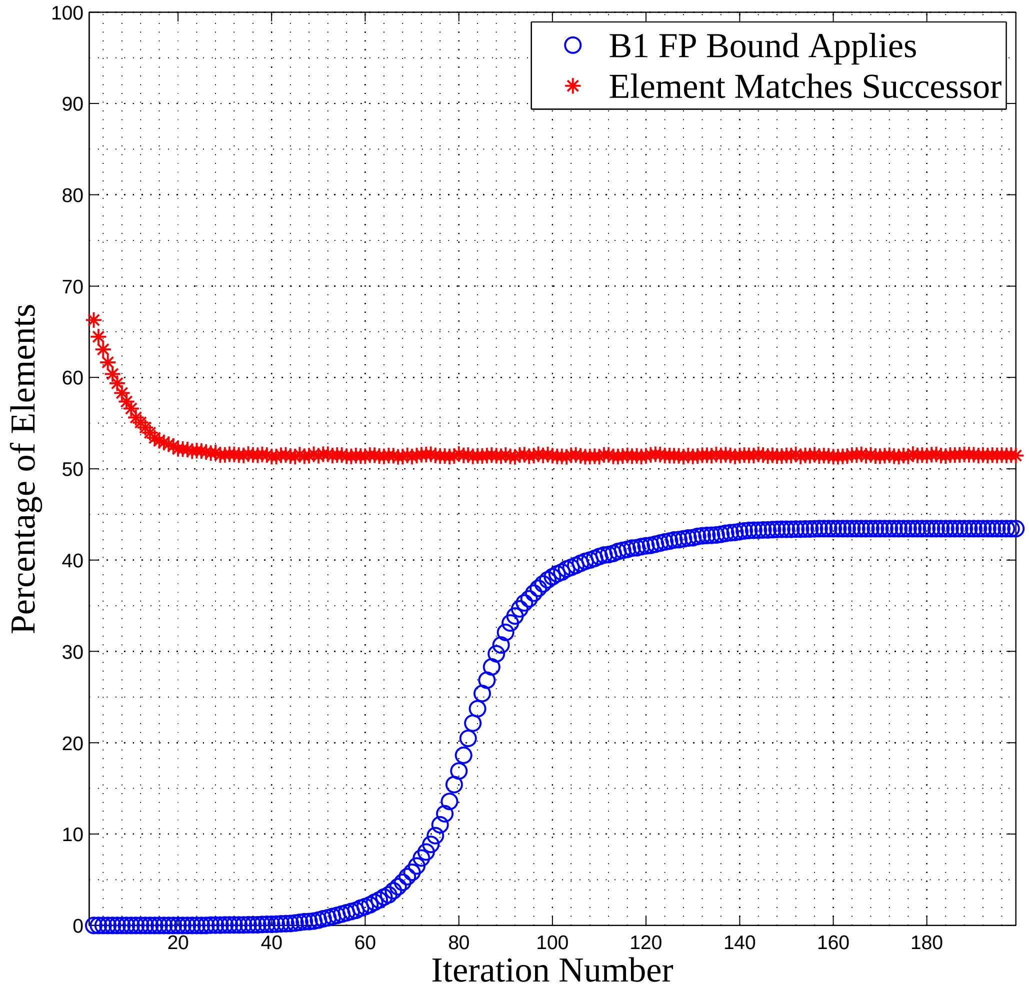
Why?

$$
\begin{aligned}
(\boldsymbol{x}_i - \boldsymbol{\pi}_i) - (\boldsymbol{x}_j - \boldsymbol{\pi}_j) \;&\leq\; \|\boldsymbol{x} - \boldsymbol{\pi}\|_1 \leq \boldsymbol{\beta} \\
\boldsymbol{x}_i - (\boldsymbol{x}_j + \boldsymbol{\beta}) \;&\leq\; \boldsymbol{\pi}_i - \boldsymbol{\pi}_j
\end{aligned}
$$

$$
\mathbf{0} < \boldsymbol{x}_i - (\boldsymbol{x}_j + \boldsymbol{\beta}) \Longrightarrow \mathbf{0} < \boldsymbol{\pi}_i - \boldsymbol{\pi}_j
$$

[Kirkland 2006]

# Applicability of Criterion



# Properties of Ranking Criterion

- Applies to any approximation, provided error bound is available
- Requires well-separated elements
- Tends to identify ranks of larger elements
- Determines partial ranking
- Top-k, bucket and exact ranking
- Easy to use with power method

# Top-k Ranking

Given:

Approximation $\boldsymbol{x}$ to PageRank $\boldsymbol{\pi}$
Permutation $\boldsymbol{P}$ so that $\tilde{\boldsymbol{x}} = \boldsymbol{P}\boldsymbol{x}$ with

$$\tilde{\boldsymbol{x}}_1 \geq \ldots \geq \tilde{\boldsymbol{x}}_n$$

Write: $\tilde{\boldsymbol{\pi}} = \boldsymbol{P}\boldsymbol{\pi}$

Suppose $\tilde{\boldsymbol{x}}_k > \tilde{\boldsymbol{x}}_{k+1} + \beta$

# Top-k Ranking

$$
\begin{array}{ccc}
\tilde{x}_1 & & \mathrm{Orank}(\tilde{\pi}_1) \leq k \\
\vdots & & \vdots \\
\tilde{x}_k & & \mathrm{Orank}(\tilde{\pi}_k) \leq k \\
& \uparrow & \\
& \beta & \\
& \downarrow & \\
\tilde{x}_{k+1} & & \mathrm{Orank}(\tilde{\pi}_{k+1}) \geq k+1 \\
\vdots & & \vdots \\
\tilde{x}_n & & \mathrm{Orank}(\tilde{\pi}_n) \geq k+1
\end{array}
$$

# Exact Ranking

Given:

Approximation $\boldsymbol{x}$ to PageRank $\boldsymbol{\pi}$

Permutation $\boldsymbol{P}$ so that $\tilde{\boldsymbol{x}} = \boldsymbol{P}\boldsymbol{x}$ with

$$\tilde{\boldsymbol{x}}_1 \geq \ldots \geq \tilde{\boldsymbol{x}}_n$$

Write: $\tilde{\boldsymbol{\pi}} = \boldsymbol{P}\boldsymbol{\pi}$

If $\tilde{\boldsymbol{x}}_{k-1} > \tilde{\boldsymbol{x}}_k + \beta$ and $\tilde{\boldsymbol{x}}_k > \tilde{\boldsymbol{x}}_{k+1} + \beta$ then

$$\mathbf{Orank}(\tilde{\boldsymbol{\pi}}_k) = \boldsymbol{k}$$

# Exact Ranking

$$
\begin{array}{llll}
\tilde{\boldsymbol{x}}_1 & & & \\
\vdots & & & \\
\tilde{\boldsymbol{x}}_{k-1} & & & \\
& \updownarrow & \boldsymbol{\beta} & \textbf{Orank}(\tilde{\pi}_k) \geq \boldsymbol{k} \\
& \tilde{\boldsymbol{x}}_k & & \\
& \updownarrow & \boldsymbol{\beta} & \textbf{Orank}(\tilde{\pi}_k) \leq \boldsymbol{k} \\
\tilde{\boldsymbol{x}}_{k+1} & & & \\
\vdots & & & \\
\tilde{\boldsymbol{x}}_n & & &
\end{array}
$$

# Bucket Ranking

Given:

Approximation $\boldsymbol{x}$ to PageRank $\boldsymbol{\pi}$

Permutation $\boldsymbol{P}$ so that $\tilde{\boldsymbol{x}} = \boldsymbol{P}\boldsymbol{x}$ with

$$\tilde{\boldsymbol{x}}_1 \geq \ldots \geq \tilde{\boldsymbol{x}}_n$$

Write: $\tilde{\boldsymbol{\pi}} = \boldsymbol{P}\boldsymbol{\pi}$

Suppose $\tilde{\boldsymbol{x}}_{k-i} > \tilde{\boldsymbol{x}}_k + \boldsymbol{\beta}$ and $\tilde{\boldsymbol{x}}_k > \tilde{\boldsymbol{x}}_{k+j} + \boldsymbol{\beta}$

$\tilde{\boldsymbol{\pi}}_k$ is in bucket of width $\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{1}$

# Bucket Ranking

$$\tilde{x}_1$$

$$\vdots$$

$$\tilde{x}_{k-i}$$

$$\updownarrow \quad \beta \qquad \text{Orank}(\tilde{\pi}_k) \geq k - i + 1$$

$$\tilde{x}_k$$

$$\updownarrow \quad \beta \qquad \text{Orank}(\tilde{\pi}_k) \leq k + j - 1$$

$$\tilde{x}_{k+j}$$

$$\vdots$$

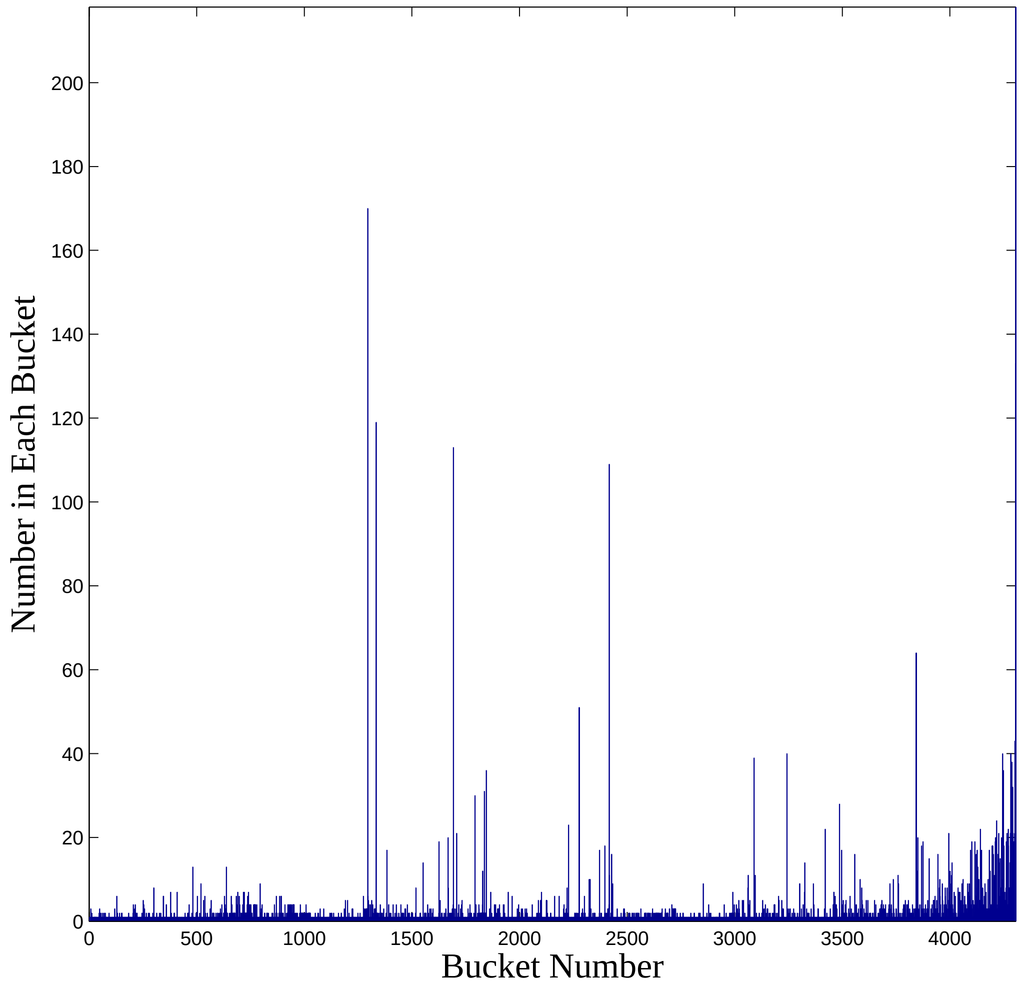
$$\tilde{x}_n$$

# Experiments

| $n$ | # buckets | 1. bucket | last bucket |
|---:|---:|---:|---:|
| 9,914 | 4,307 | 1 | 7% |
| 3,148,440 | 34,911 | 1 | 95% |

| $n$ | exact rank | exact top 100 | lowest rank |
|---:|---:|---:|---:|
| 9,914 | 32% | 79 | 9,215 |
| 3,148,440 | 0.76% | 100 | 151,794 |

# Buckets for Small Matrix



# Power Method Ranking

Simple error bound:

$$\|x^{(k)} - \pi\|_1 \leq \underbrace{2\alpha^k}_{\beta}$$

Simple ranking criterion:

If $x_i^{(k)} > x_j^{(k)} + 2\alpha^k$ then $\pi_i > \pi_j$

But: $2\alpha^k$ is too pessimistic (not tight enough)

# Power Method Ranking

Tighter error bound:

$$\|x^{(k)} - \pi\|_1 \leq \underbrace{\frac{\alpha}{1-\alpha}\|x^{(k)} - x^{(k-1)}\|_1}_{\beta}$$

More effective ranking criterion:

If $x_i^{(k)} > x_j^{(k)} + \beta$ then $\pi_i > \pi_j$

# Floating Point Ranking

If $x_i^{(k)} > x_j^{(k)} + \beta$ then $\pi_i > \pi_j$

$$\beta = \frac{\alpha}{1-\alpha} \|x^{(k)} - x^{(k-1)}\|_1 + e$$

$e$ is round off error from single matvec

IEEE double precision floating point arithmetic:

$$e \approx cm10^{-16}$$

$m \approx$ max # links into any web page

# Expensive Implementation

To prevent accumulation of round off

- Explicit normalization of iterates
  $\boldsymbol{x}^{(k+1)} = \boldsymbol{x}^{(k+1)} / \|\boldsymbol{x}^{(k+1)}\|_1$
- Compute norms, inner products, matvecs
  with compensated summation
- Limited by round off error from single matvec
- Analysis for matrix dimensions $n < 10^{14}$
  in IEEE arithmetic ($\epsilon \approx 10^{-16}$)

# Difficulties for XLARGE Problems

- Catastrophic cancellation when computing

$$\beta = \frac{\alpha}{1-\alpha}\|x^{(k)} - x^{(k-1)}\|_1 + e$$

- Bound $\beta$ dominated by round off $e$
- Compensated summation insufficient to reduce higher order round off $\mathcal{O}(n\epsilon^2)$
- Doubly compensated summation too expensive: $\mathcal{O}(n \log n)$ flops

# Possible Remedies

- Lump dangling nodes [Ipsen & Selee 2006]
  Web pages w/o outlinks:
  pdf & image files, protected pages, web frontier
  Up to 50%-80% of all web pages
- Remove unreferenced web pages
- Use faster converging method
  Then 1 power method iteration for ranking
- Relative ranking criteria?

# Summary

- Google orders web pages according to: PageRank and hypertext analysis
- PageRank = left eigenvector of $G$

$$G = \alpha S + (1 - \alpha)\mathbb{1}v^T$$

- Power method: simple, robust, accurate
- Convergence rate depends on $\alpha$ but not on matrix dimension
- Criterion for ordinal ranking
- Round off serious for XLarge problems

# User-Friendly Resources

- Rebecca Wills:
  *Google's PageRank: The Math Behind the Search Engine*
  Mathematical Intelligencer, 2006
- Amy Langville & Carl Meyer:
  *Google's PageRank and Beyond*
  *The Science of Search Engine Rankings*
  Princeton University Press, 2006
- Amy Langville & Carl Meyer:
  Broadcast of On-Air Interview, November 2006
  Carl Meyer's web page