Numerical Reliability of Randomized Algorithms

Inner Product – Two Norm

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Randomized Matrix Multiplication

Sarlós 2006
Drineas, Kannan & Mahoney 2006
Belabbas & Wolfe 2008

Goal:

Algorithm behaviour for moderate matrix dimensions
Numerical properties of algorithms

Outline

Randomized inner product – squared two norm
Relative error due to randomization
Repeated sampling of same elements
“Stability” of algorithm

Randomized Inner Product – Squared Two Norm

[Drineas, Kannan & Mahoney 2006]

Input: real vector $a = \begin{pmatrix} a_1 & \dots & a_n \end{pmatrix}^T$
probabilities $p_k > 0$, $\sum_{k=1}^n p_k = 1$
number c where $1 \le c \le n$

Output: Approximation X to $a^T a$
from c randomly sampled elements a_k

$X = 0$
for $t = 1 : c$ **do**
Sample k_t from $\{1, \dots, n\}$ with probability p_{k_t}
independently and with replacement
$X = X + \frac{a_{k_t}^2}{c\, p_{k_t}}$
end for

Properties

[Drineas, Kannan & Mahoney 2006]

Unbiased estimator

$$E[X] = a^T a$$

Uniform probabilities: $p_k = 1/n$, $1 \leq k \leq n$

Absolute error bound

For every $\delta > 0$ with probability at least $1 - \delta$

$$\left| a^T a - X \right| < \frac{n \, \|a\|_\infty^2}{\sqrt{c}} \sqrt{8 \ln(2/\delta)}$$

Relative Error due to Randomization

Relative Error Bound

For every $\delta > 0$ with probability at least $1 - \delta$

$$\frac{|X - a^T a|}{a^T a} < \epsilon$$

where

$$\epsilon \geq \frac{1}{\sqrt{c\,\delta}} \sqrt{\sum_{k=1}^{n} \frac{a_k^4}{p_k \, \|a\|_2^4} - 1}$$

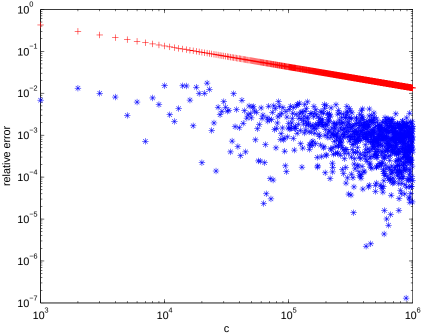
Proof: Chebyshev inequality

Uniform probabilities $p_k = 1/n$

$$\epsilon \geq \frac{1}{\sqrt{c\,\delta}} \sqrt{n \left(\frac{\|a\|_4}{\|a\|_2} \right)^4 - 1}$$

Relative Error for Uniform Probabilities

$n = 10^6$, a_k are independent uniform $[0, 1]$



Relative errors $|X - a^T a|/a^T a$ for every c

Chebyshev bound with probability .99

Relative Error for Uniform Probabilities

Uniform vectors

a_k *iid uniform* $[0, 1]$, $n = 10^6$

Relative error: $10^{-2} - 10^{-1}$

Weakly graded vectors

$a = \begin{pmatrix} 1 & 2 & \dots & n \end{pmatrix}^T$

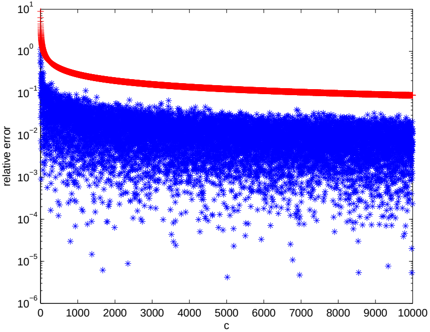
With probability $1 - \delta$: *Relative error* $\geq .8/\sqrt{\delta\, c}$

With 99 percent probability:

Relative error $\approx 10^{-8}$ *for* $c \geq 10^{20}$

Weakly Graded Vectors

$a = \begin{pmatrix} 1 & 2 & \dots & n \end{pmatrix}^T$, $n = 10^4$

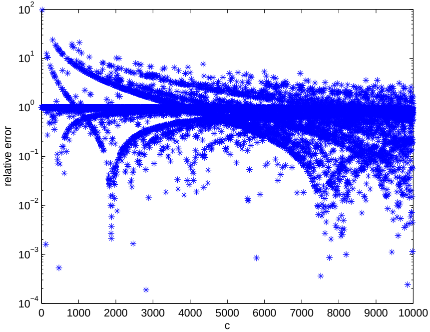


Relative errors $|X - a^T a|/a^T a$ for every c

Chebyshev bound with probability .99

Strongly Graded Vectors

$a = \begin{pmatrix} 1 & 2^{-1} & \dots & 2^{-n+1} \end{pmatrix}^T$, $n = 10^4$



Relative errors $|X - a^T a| / a^T a$ for every c

Relative error $\geq \sqrt{.6n - 1} / \sqrt{\delta\, c}$ grows with n

Non-Uniform Probabilities

Sample a_k with probability $p_k = |a_k|/\|a\|_1$

- For every $\delta > 0$ with probability at least $1 - \delta$

  $$\frac{|X - a^T a|}{a^T a} < \epsilon$$

  where

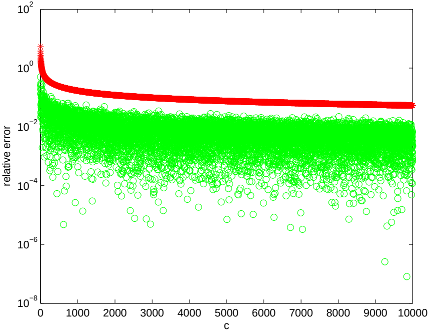
  $$\epsilon \geq \frac{1}{\sqrt{c\,\delta}} \sqrt{\frac{\|a\|_1\,\|a\|_3^3}{\|a\|_2^4} - 1}$$

  Smaller than relative error for uniform probabilities

- Weakly and strongly graded vectors
  Relative error $\geq .3/\sqrt{\delta\, c}$ independent of n

Strongly Graded Vectors

$a = \begin{pmatrix} 1 & 2^{-1} & \dots & 2^{-n+1} \end{pmatrix}^T$, $n = 10^4$
non uniform probabilities $p_k = |a_k| / \|a\|_1$



Relative errors $|X - a^T a| / a^T a$ for every c
Chebyshev bound with probability .99

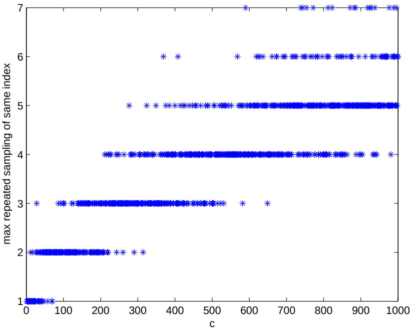
Relative Errors: Summary

- Moderate dimensions
  For $n \leq 10^6$: relative error $\approx 10^{-2} - 10^{-1}$
  Output of algorithm has 1-2 correct decimal digits
- Larger dimensions
  For relative error of 10^{-8} need dimension $n \geq 10^{20}$
- Uniformly distributed and weakly graded vectors
  Uniform probabilities suffice
- Strongly graded vectors
  Need non-uniform probabilities
- Probability bounds
  Hoeffding's bound is tighter by only factor of 10 compared to Chebyshev bound

Repeated Sampling of Same Elements

Maximal Number of Times Same Element Is Sampled

$n = 10^3$, a_k iid uniform $[0,1]$, uniform probabilities



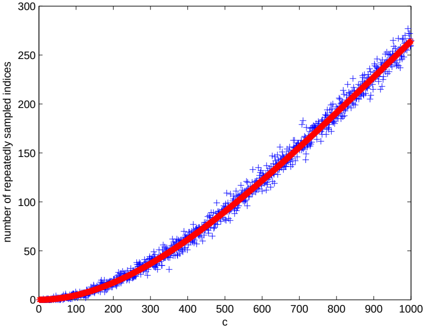
Repeated sampling increases with c

Elements that are Repeatedly Sampled

Expected value of # distinct elements sampled more than once

$$n\left(1-\left(1-\frac{1}{n}\right)^{c-1}\left(1+\frac{c-1}{n}\right)\right) \approx n-(n+c)e^{-c/n}$$

$$\approx .27n \qquad \text{for } c=n$$

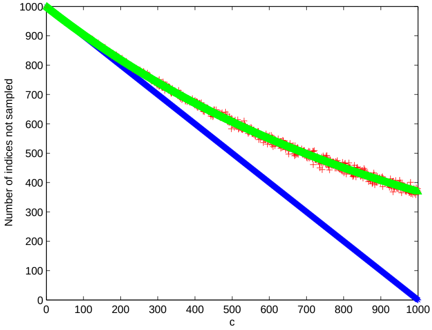


Elements that are Never Sampled

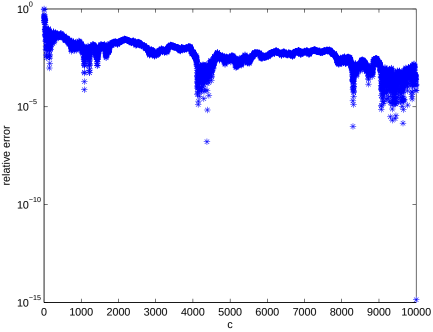
Expected value of # elements never sampled

$$n\left(1-\frac{1}{n}\right)^{c} \approx n\,e^{-c/n}$$

$$\approx .37n \qquad \text{for } c = n$$



No Repeated Sampling



Relative errors $|X - a^T a|/|a^T a|$ for every c

Relative errors still around $10^{-2} - 10^{-1}$

Repeated Sampling

Uniform probabilities

- Number of times an element can be sampled increases with c
- About 27% elements sampled more than once
- About 37% elements never sampled
- Repeated sampling does not seem to hurt accuracy

Non-uniform probabilities

- Preliminary conjecture: repeated sampling occurs
  at same rate as for uniform probabilities

“Stability” of Randomized Algorithm

What is Stability?

- Stability of deterministic algorithms:

  How does a perturbation of the input change the output of the algorithm?

- Difficulty with randomized algorithms:
  We don't know the output with certainty

- Exception:
  Constant vector $\quad a_k = \alpha, \quad 1 \leq k \leq n$

  Uniform probabilities:

$$X = \underbrace{\frac{n}{c}\alpha^2 + \cdots + \frac{n}{c}\alpha^2}_{c} = n\alpha^2 = a^T a$$

  Randomized algorithm gives exact result for any c

Stability of Randomized Algorithm

- Relative perturbations of constant vector

$$\tilde{a}_k = \alpha\,(1 + \epsilon\,\rho_k)$$

$0 < \epsilon \ll 1$, ρ_k are iid random variables

- Perturbed approximation

$$\tilde{X} = \frac{n}{c}\left(\tilde{a}_{k_1}^2 + \cdots + \tilde{a}_{k_c}^2\right)$$

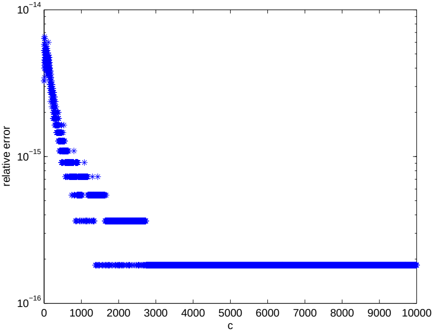
- Algorithm is numerically stable if

$$\underbrace{\left|\frac{\tilde{X} - n\alpha^2}{n\alpha^2}\right|}_{\textit{forward error}} = \mathcal{O}(\epsilon)$$

Forward Errors

$\alpha = 1, \; n = 10^4$

Perturbations: $\epsilon = 10^{-14}$, $\quad \rho_k$ iid uniform $[0, 1]$



Forward errors $(\tilde{X} - n\alpha^2)/(n\alpha^2)$ for every c

Forward errors bounded by $\epsilon \quad \Rightarrow$ algorithm stable

Expected Value of Forward Error

- First and second moments

$$E_\rho[\rho_k] = \mu_1 \qquad E_\rho[\rho_k^2] = \mu_2$$

- Expected value of forward error

$$E_\rho\left[\frac{\tilde{X} - n\alpha^2}{n\alpha^2}\right] = 2\epsilon\,\mu_1 + \epsilon^2\,\mu_2$$

- If perturbations ρ_k are iid uniform $[\beta_1, \beta_2]$ then

$$E_\rho\left[\frac{\tilde{X} - n\alpha^2}{n\alpha^2}\right] = \epsilon\,(\beta_1 + \beta_2) + \frac{\epsilon^2}{3}\,\left(\beta_1^2 + \beta_1\beta_2 + \beta_2^2\right)$$

Expected value of forward error is $\mathcal{O}(\epsilon)$

How Close is Forward Error To Expected Value?

- Perturbations ρ_k are iid uniform $[\beta_1, \beta_2]$
- Probability that

$$\left| \frac{\tilde{X} - n\alpha^2}{n\alpha^2} - E_\rho \left[\frac{\tilde{X} - n\alpha^2}{n\alpha^2} \right] \right| < \tau$$

is at least

$$1 - 2\ \exp\left(\frac{-\tau^2\ c}{2\ \left(\epsilon\,(\beta_2 - \beta_1) + \epsilon^2\ \max\{\beta_1^2, \beta_2^2\}\ \right)^2} \right)$$

Proof: Azuma's inequality

Bound on Forward Error

- Perturbations ρ_k are iid uniform $[\beta_1, \beta_2]$

$$\left|\frac{\tilde{X} - n\alpha^2}{n\alpha^2}\right| < \epsilon\ (1 + |\beta_1 + \beta_2|) + \frac{\epsilon^2}{3}\ \left|\beta_1^2 + \beta_1\beta_2 + \beta_2^2\right|$$

holds with probability at least $1 - \delta$ for

$$c \geq 2 \ln\left(\frac{2}{\delta}\right)\ \left((\beta_2 - \beta_1) + \epsilon\ \max\{\beta_1^2, \beta_2^2\}\right)^2$$

- Perturbations ρ_k are iid uniform $[0, 1]$

$$\left|\frac{\tilde{X} - n\alpha^2}{n\alpha^2}\right| < 3\,\epsilon$$

holds with probability at least .99 for $c \geq 22$

Summary

- Randomized algorithm for inner product $a^T a$
  from [Drineas, Kannan & Mahoney 2006]
- Low relative accuracy
  1-2 correct decimal digits for dimensions $n \leq 10^6$
- Repeated sampling of elements occurs frequently
  but does not seem to hurt accuracy
- Preliminary definition of numerical stability

  Change in output when constant vector
  perturbed by iid random variables

- Randomized algorithm is stable w.r.t. perturbations
  by iid uniform $[\beta_1, \beta_2]$ variables